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# Construction of the Jordan basis for the Baker map

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The Jordan canonical form basis states for an invertible chaotic map, the Baker map, are constructed. A straightforwardly obtained recursion formula is presented for construction of the Jordan states and of the spectral decomposition of the Frobenius–Perron evolution operator. Comparison of this method with earlier, subdynamics techniques demonstrates that it is much more direct and simpler. The physical significance of the Jordan states is approached from the point of view of an entropy evolution equation. The method is also applied to the Bernoulli map, yielding its eigenstates more straightforwardly than done previously. 
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**A reconciliation between the time reversal invariance of microscopic dynamics and macroscopic irreversibility has attracted interest for many years. A fresh perspective has been developed recently within the context of discrete chaotic maps. These dynamical systems permit detailed analysis in which all necessary mathematical objects can be explicitly constructed. This constructibility is especially important in this context because the evolution operator for these chaotic maps is not self-adjoint, which means it does not have simple eigenstates and eigenvalues. Instead, a more complicated Jordan basis must be constructed that contains both right-sided and left-sided states. A spectral decomposition of the evolution operator is the desired goal because it lays bare the connection between spectral properties and irreversibility. In this paper, the Baker map is studied. This two-dimensional chaotic map is one of the simplest, invertible discrete analogues to a reversible continuous dynamical system. Previous work on this type of map has involved quite complicated and lengthy analysis utilizing resolvent operator algebra. Here, the method of construction of the Jordan states for the Baker map is much simpler than the previous ones.**

## I. INTRODUCTION

An intrinsic basis for the reconciliation between the time reversal invariance of conservative Hamiltonian dynamics and irreversible macroscopic observations has been the subject of research for some time.[1a] A possible resolution has been proposed recently[1b–1d] and constructively developed[2] within the context of chaotic iterated maps. This work has its foundation in earlier work on Pollicott–Ruelle resonances in axiom-*A* systems.[3] The reconciliation has its roots in the differences between the behavior of pointwise, trajectory solutions and smooth, extended density solutions.[4] We recently explored this difference from the perspective of the instability of trajectory solutions for chaotic maps.[5] This earlier paper is referred to as **I** in the following presentation. In the present paper, we answer several questions left open in **I**, and raise others. Our work follows closely the lead of Hasegawa *et al.*[2a,2b] and of Antoniou and Tasaki,[2c] which in turn is related to the fundamental work of Gaspard.[1b–1d] As in all of the earlier work, the Frobenius–Perron (F–P) operator associated with an iterated map is the focus of our studies. Its Jordan state basis and its spectral decomposition are of central interest. In this paper, a remarkably straightforward recursion method is presented that permits construction of the Jordan basis as well as the spectral decomposition of the F–P operator. The simplest paradigms for these constructions are the Bernoulli map and the Baker map.[2,4,6] The Baker map is a two-dimensional, measure preserving, invertible chaotic map. It is a discrete analogue of a continuous in time dynamical system.

Dynamically, the Baker map is unstable in one degree of freedom and stable in the other. When integration over the stable degree of freedom is performed, the contracted description is precisely that for the Bernoulli map, at least from the perspective of the F–P equation.[4–7] The eigenfunctions for the Bernoulli F–P operator are the Bernoulli polynomials.[2,5,8] Because the F–P operator is not self-adjoint, these eigenfunctions are right-sided only, and a different class of functions makes up the left-sided eigenfunctions.[2] These left-sided eigenfunctions have been constructed for the Bernoulli map.[2a,2b] Here, we present an alternative treatment, adumbrated in a paper by Gaspard.[8] In doing so, we are able to shorten considerably the construction of these eigenfunctions as compared with the subdynamics method used in Ref. 2.

As will be emphasized below, the left-sided eigenfunctions are meaningful only in a formal sense.[2] The adjoint of the F–P operator is called the Koopman (K) operator,[4] so that the left-sided eigenfunctions of the F–P operator are right-sided eigenfunctions of the K operator. In **I**, we observed that the K operator for the Bernoulli map is identical with the F–P operator for a certain iterated function system (IFS).[5,9] Thus, the eigenfunctions for the IFS F–P operator, missing in **I**, now can be explicitly exhibited. These eigenfunctions together with the eigenfunctions for the Bernoulli F–P operator enable us to construct Jordan states for the Baker F–P operator. This enables us to obtain the spectral decomposition for the Baker F–P operator in a very straightforward fashion. The fundamental recursion formula, Eq.

(60), is an alternative to the recursion formulas of the subdynamics constructions.[2]

The spectrum for the Bernoulli F–P operator, which is identical to the spectrum for the IFS F–P operator, is nondegenerate. Consequently, the Jordan canonical form is purely diagonal and the Jordan basis is comprised of genuine eigenstates.[10] However, in the case of the Baker F–P operator, the spectrum is degenerate, matrix representations are nontrivially nondiagonalizable, and eigenstates are no longer possible (except for one state in each degenerate Jordan block). In this case, a nontrivial Jordan basis must be constructed instead, in order to obtain a canonical form. Degeneracy *per se* is not sufficient to require the nontrivial Jordan basis, and this means that the Baker map is different from a genuinely Hamiltonian system for which degeneracy does not necessitate a nontrivial Jordan basis.

In **I**, the iterated map evolution of periodic Gaussian densities was studied. For the Bernoulli F–P equation, it was shown that the limit of vanishing standard deviations and the limit of long-time evolution do not commute. This feature of the dynamics demonstrates the intrinsic instability of pointwise, trajectory solutions for chaotic maps. For the IFS F–P equation (Bernoulli K equation), these limits are interchangeable and identical, thereby exhibiting the stability of pointwise, trajectory solutions in this case. These facts enable us to see that for the Baker F–P equation a mixture of both behaviors occurs. A consequence of this observation is the apparent paradox that for the Baker F–P equation, Gibbs entropy is an invariant,[4] whereas Jordan basis expansions appear to irreversibly evolve towards the invariant density, which has maximal entropy.

The remainder of this paper is organized as follows. In Sec. II, we review the properties of the Bernoulli F–P equation, its associated Koopman operator, construct the left-sided eigenfunctions, and give the spectral decomposition of the F–P operator. The left-sided eigenfunctions are seen to be meaningful in a formal sense. In Sec. III, we apply these results to the Baker map. The spectrum and a construction of the formal Jordan basis as well as the spectral decomposition of the F–P operator, missing in **I**, are presented. The fundamental recursion formula for these constructions is derived here in a remarkably direct way compared to what has been done previously.[2] In Sec. IV, we discuss several other recursion formulas and related combinatorial identities. These combinatorial identities involve Bernoulli numbers and may be of broader interest. In Sec. V, we discuss the entropy evolution equation for the Baker map. It is this issue that is at the heart of putative intrinsic irreversibility of solutions to invertible chaotic maps.

## II. EIGENSTATES FOR THE BERNOULLI MAP

Let a generic map on $[0, 1]$ be denoted by $M(x)$, so that

$$x_{n+1} = M(x_n). \tag{1}$$

The associated F–P equation is[2,4–6]

$$\begin{aligned} P_M F(x) &= \int_0^1 dy\ \ \delta(x - M(y)) f(y) \\ &= \sum_{\{M^{-1}(x)\}} \frac{f(M^{-1}(x))}{|M'(M^{-1}(x))|}, \end{aligned} \tag{2}$$

in which $\{M^{-1}(x)\}$ denotes the set of inverse images of $x$, and $M'$ is the derivative of $M$. Here $P_M$ is the F–P operator and $f(x)$ is an arbitrary function. For the Bernoulli map,[2,4,5]

$$M(x) = D_B(x) = 2x \text{ MOD } 1, \tag{3}$$

and the F–P equation is[2,4,5]

$$P_B f(x) = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(\frac{x}{2} + \frac{1}{2}\right). \tag{4}$$

The right-sided eigenfunction–eigenvalue equation is[2,4,5]

$$P_B B_m(x) = \frac{1}{2^m} B_m(x), \tag{5}$$

in which $B_m$ is a Bernoulli polynomial. A derivation of this result appears in Refs. 2 and 5, and a concise review of the properties of Bernoulli polynomials appears in appendix A of Ref. 5. It is convenient in the following to introduce additional scale factors and to define the right-sided eigenfunctions, $R_m(x)$, by

$$R_m(x) = (-1)^m \sqrt{(2m+1)}\, \frac{(2m)!}{(m!)^2} B_m(x). \tag{6}$$

A natural $L_2$ basis for the unit interval[11] is provided by the modified Legendre polynomials.[2] Let $P_n(x)$ denote the $n$th Legendre polynomial, which is defined on the interval $[-1,1]$, and define the modified Legendre polynomial, $\phi_n(x)$, on the unit interval, $[0, 1]$, by[2]

$$\phi_n(x) = \sqrt{(2n+1)} P_n(1-2x). \tag{7}$$

These functions satisfy orthonormality

$$\int_0^1 dx\, \phi_n(x) \phi_m(x) = \delta_{nm}, \tag{8}$$

and completeness

$$\sum_{n=0}^{\infty} \phi_n(x) \phi_n(x') = \delta(x - x'). \tag{9}$$

The Rodriguez formula for Legendre polynomials implies[2]

$$\phi_n(x) = \frac{\sqrt{(2n+1)}}{n!} \frac{d^n}{dx^n} (x(1-x))^n. \tag{10}$$

In Appendix A, we prove that $R_m(x)$ may be expanded in terms of the $\phi_n(x)$'s according to the formula

$$R_m(x) = \sum_{n=0}^{m} R_{mn} \phi_n(x) \tag{11}$$

in which the matrix $R_{mn}$ is defined, for $n \leq m$ and $m+n$, even, by

$$R_{mn}=\sqrt{(2m+1)(2n+1)}\,\frac{(2m)!}{m!}\times\sum_{r=0}^{m-n}B_{m-n-r}\,\frac{(n+r)!}{r!(m-n-r)!(2n+r+1)!}. \quad (12)$$

in which $B_{m-n-r}$ is a Bernoulli number.[5,12] $R_{mn}$ vanishes for $m+n$ odd and for $m<n$, i.e., $\mathbf{R}$ is a lower triangular matrix.

For arbitrary functions $f(x)$ and $g(x)$ the adjoint of the F–P operator $P_M$ is called the Koopman operator,[4] is denoted by $K_M$, and is defined by

$$\int_0^1 dx\; f(x)P_Mg(x)=\int_0^1 dx\; g(x)K_Mf(x). \quad (13)$$

The left-sided eigenfunctions of $P_M$ are right-sided eigenfunctions of $K_M$. In **I**, we showed that for the Bernoulli map, the Koopman operator, $K_B$, is equal to the F–P operator for a certain iterated function system, $P_{\mathrm{IFS}}$. This IFS is based on the stochastic iteration of two maps,[5] each with probability $\frac{1}{2}$,

$$M_1(x)=\frac{x}{2}, \quad (14)$$

$$M_2(x)=\frac{x}{2}+\frac{1}{2}. \quad (15)$$

The F–P equation for this IFS is

$$P_{\mathrm{IFS}}f(x)=f(2x)\Theta(\tfrac{1}{2}-x)+f(2x-1)\Theta(x-\tfrac{1}{2}) \quad (16)$$

in which $\Theta(x)$ is the Heaviside theta function.

The eigenfunction–eigenvalue equation for the left-sided eigenfunctions of $P_B$ is

$$K_BL_m(x)=\lambda_mL_m(x). \quad (17)$$

It is known[2] that these functions, together with the right-sided eigenfunctions of $P_B$, satisfy biorthonormality

$$\int_0^1 dx\; L_n(x)R_m(x)=\delta_{nm} \quad (18)$$

and completeness

$$\sum_{n=0}^{\infty} L_n(x)R_n(x')=\delta(x-x'). \quad (19)$$

The definition of the adjoint operator in Eq. (13) permits determination of the eigenvalues in Eq. (17). Clearly

$$\int_0^1 dx\; L_m(x)P_BR_n(x)=\int_0^1 dx\; R_n(x)K_BL_m(x) \quad (20)$$

implies

$$\left(\frac{1}{2^n}-\lambda_m\right)\int_0^1 dx\; L_m(x)R_n(x)=0. \quad (21)$$

This implies Eq. (18) and

$$\lambda_m=\frac{1}{2^m}. \quad (22)$$

Thus, taking the adjoint is isospectral, as is well known.

An analogue to the expansion in Eq. (11) exists and is

$$L_m(x)=\sum_{n=m}^{\infty} L_{mn}\phi_n(x). \quad (23)$$

The inverse of Eq. (11) is

$$\phi_n(x)=\sum_{m'=0}^{n}(R^{-1})_{nm'}R_{m'}(x). \quad (24)$$

Therefore (the repeated index summation convention is used),

$$L_{mn}=\int_0^1 dx\; L_m(x)\phi_n(x)=\int_0^1 dx\; L_m(x)(R^{-1})_{nm'}R_{m'}(x)=(R^{-1})_{nm}, \quad (25)$$

wherein we have used Eqs. (8) and (18). This means

$$\mathbf{L}=(\mathbf{R}^{\dagger})^{-1} \quad (26)$$

in which $\mathbf{R}^{\dagger}$ denotes the adjoint of $\mathbf{R}$. Thus, $\mathbf{L}$ is upper triangular. In Appendix B, we derive the explicit formula for the matrix elements $L_{mn}$ given by[2]

$$L_{mn}=\sqrt{\frac{2n+1}{2m+1}}\frac{(m+n-1)!}{(2m-1)!(n-m+1)!}, \quad (27)$$

which is valid for $1\leqslant m\leqslant n$ and $m+n$ even. For $m>n$, the matrix elements vanish, as well as for $m+n$ odd.

There is a major difference between the right-sided and left-sided eigenfunctions of $P_B$. The $R_m(x)$'s are expandable in terms of the $\phi_n(x)$'s using a finite number of coefficients $R_{mn}$ with $n\leqslant m$. The $L_m(x)$'s, on the other hand, require infinitely many coefficients when expanded in terms of the $\phi_n(x)$'s, and these coefficients [see Eq. (27)] diverge with increasing $n$.[2] Thus, these left-sided eigenfunction expressions are meaningful only in a formal sense. Much like the Dirac delta function and its derivatives, which are meaningful in the sense of Schwartz distributions, these formal expressions are meaningful upon integration.[2]

Let $f(x)$ be a polynomial of degree $N$. The orthonormal $\phi_n(x)$ basis permits the expansion

$$f(x)=\sum_{n=0}^{N} d_n\phi_n(x), \quad (28)$$

where

$$d_n=\int_0^1 dx\; f(x)\phi_n(x). \quad (29)$$

Similarly, we have expansions in terms of the $L_m(x)$'s

$$f(x)=\sum_{m=0}^{\infty} c_mL_m(x). \quad (30)$$

Notice the infinite range for the summation here as compared with Eq. (28). The proof that $d_n=0$ for all $n \geqslant N$ utilizes the Rodriguez identity, Eq. (10), and integration by parts in Eq. (29).

Let $K_B^p$ denote the $p$th iteration of $K_B$ and write, according to Eqs. (17) and (22),

$$K_B^p f(x)=\sum_{m=0}^{\infty} c_m \frac{1}{2^{pm}} L_m(x). \tag{31}$$

Thus, while formal, this expression conveys the nature of the approach to equilibrium. Equation (27) shows that the convergence here is delicate. This has been carefully discussed elsewhere.[2,8]

To motivate the construction of the spectral decomposition of the Baker F–P operator given in the next section, the spectral decomposition for the Bernoulli F–P operator will be given here. Let us introduce the Dirac notation

$$|m\rangle = R_m(x) \quad \text{and} \quad \langle m| = L_m(x), \tag{32}$$

so that we may give the Bernoulli F–P operator the explicit spectral decomposition

$$P_B=\sum_m |m\rangle \frac{1}{2^m} \langle m|. \tag{33}$$

Together with biorthonormality, Eq. (18), and scaling, Eq. (6), this yields Eq. (5), the eigenstate equation for the Bernoulli polynomials. It also yields the corresponding left-sided eigenstate equation equivalent to Eq. (17). The simplicity of this result is a consequence of the nondegeneracy of the Bernoulli F–P operator spectrum. In the case of the Baker F–P operator, the corresponding result is more complicated because the spectrum is degenerate and the Baker map requires the nontrivial Jordan canonical form.

## III. JORDAN BASIS FOR THE BAKER MAP

The Baker map[2,4–7] is defined by

$$(x_{n+1},y_{n+1})=\left(2x_n, \frac{y_n}{2}\right)\Theta\left(\frac{1}{2}-x_n\right) + \left(2x_n-1, \frac{y_n}{2}+\frac{1}{2}\right)\Theta\left(x_n-\frac{1}{2}\right). \tag{34}$$

This map is area preserving and invertible, i.e., reversible. The F–P equation is

$$\begin{aligned} P_B f(x,y) &= \int_0^{1/2} dx' \int_0^1 dy' \ \delta(x-2x')\,\delta\left(y-\frac{y'}{2}\right) \\ &\quad \times f(x',y') + \int_{1/2}^1 dx' \int_0^1 dy' \ \delta(x-(2x'-1)) \\ &\quad \times \delta\left(y-\left(\frac{y'}{2}+\frac{1}{2}\right)\right) f(x',y') \\ &= f\left(\frac{x}{2},2y\right)\Theta\left(\frac{1}{2}-y\right) \\ &\quad + f\left(\frac{x}{2}+\frac{1}{2},2y-1\right)\Theta\left(y-\frac{1}{2}\right). \end{aligned} \tag{35}$$

As was shown in **I**, a two-dimensional periodic Gaussian distribution

$$\begin{aligned} f_{\alpha,\beta}(x,y) &= \frac{1}{2\pi\alpha\beta}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \exp\left[-\frac{(x-x_0+n)^2}{2\alpha^2}\right] \\ &\quad \times \exp\left[-\frac{(y-y_0+m)^2}{2\beta^2}\right] \end{aligned} \tag{36}$$

experiences a doubling of the standard deviation $\alpha$ and a halving of the standard deviation $\beta$ upon application of $P_{\text{Baker}}$. The point of emphasis in **I** was that the limits $\alpha \to 0$ and $n\to\infty$, where $n$ is the number of $P_{\text{Baker}}$ iterations, do not commute, whereas the limits $\beta\to 0$ and $n\to\infty$ do commute. The former case is identical with the corresponding behavior for the Bernoulli F–P equation, and the latter case is identical with the behavior of the IFS F–P equation (or equivalently, the Bernoulli K equation). Moreover, if $g(x)$ is defined by

$$g(x)=\int_0^1 dy\ f(x,y), \tag{37}$$

then Eq. (35) implies that $g(x)$ satisfies Eq. (4), the Bernoulli F–P equation;[4,6,7] whereas after many iterations of $P_{\text{Baker}}$, the $x$ distribution becomes asymptotically uniform and the residual $y$ behavior is indistinguishable (substitute $y$ for $x$) from the IFS behavior defined by Eqs. (14)–(16).

For these reasons, it is natural to suspect that a basis for the Baker F–P operator may be developed from products of $R_m(x)$'s and $L_n(y)$'s, at least in a Schwartzian sense. Specifically, the right-sided Jordan basis states for $P_{\text{Baker}}$ are expanded in terms of (in Dirac notation)

$$|m,n\rangle = R_m(x)L_n(y), \tag{38}$$

and the left-sided Jordan states are expanded in terms of the adjoints

$$\langle j,k| = L_j(x)R_k(y). \tag{39}$$

From Eqs. (18) and (19), we have biorthonormality

$$\langle j,k|m,n\rangle = \delta_{jm}\delta_{kn} \tag{40}$$

and completeness

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} R_m(x')L_n(y')L_m(x)R_n(y)=\delta(x'-x)\,\delta(y'-y). \tag{41}$$

The presence of the $L$'s makes these Schwartzian expressions, and in Appendix C we prove that for $j\leqslant m$ and $n\leqslant k$ (otherwise it vanishes)

$$\begin{aligned}\langle j,k|P_{\mathrm{Baker}}|m,n\rangle &= \tfrac{1}{2}[1+(-1)^{m+j+k+n}](-1)^{j+n+m+k}\\ &\quad\times\sqrt{\frac{(2m+1)(2k+1)}{(2j+1)(2n+1)}}\frac{(2m)!(2k)!}{(2j)!(2n)!}\,\frac{j!n!}{m!k!(m-j+1)!(k-n+1)!}\,\frac{1}{2^{j+n-2}}\\ &\quad\times(B_{m-j+1}(\tfrac{1}{2})-B_{m-j+1}(0))(B_{k-n+1}(\tfrac{1}{2})-B_{k-n+1}(0)).\end{aligned} \tag{42}$$

The $j=m$ and $k=n$ matrix elements are especially simple:[2]

$$\langle m,n|P_{\mathrm{Baker}}|m,n\rangle=\frac{1}{2^{m+n}} \tag{43}$$

If the matrix in Eq. (42) were purely diagonal, which it is not, these simple powers would be the eigenvalues automatically. Clearly, for $m+n=p$, they are $p+1$ fold degenerate. If either $j=m$ but $k\neq n$, or $k=n$ but $j\neq m$, then Eq. (42) contains, in the first instance, the vanishing factor

$$\begin{aligned}&(B_1(\tfrac{1}{2})-B_1(0))\tfrac{1}{2}[1+(-1)^{k+n}]\\ &\quad\times(B_{k-n+1}(\tfrac{1}{2})-B_{k-n+1}(0))=0,\end{aligned} \tag{44}$$

because the first factor is $\frac{1}{2}$, the second factor is 1 if $k+n$ is even and is 0 if $k+n$ is odd, and the third factor vanishes since $B_q(\frac{1}{2})=B_q(0)=0$ for all odd $q>1$. These last two facts follow, respectively, from Eq. (A4) and the fact that all Bernoulli numbers [recall that $B_q(0)=B_q$], $B_q$, with odd index $q>1$ vanish.[5,12] Thus, for all $k\neq n$ and for all $j\neq m$ we have[2]

$$\langle m,k|P_{\mathrm{Baker}}|m,n\rangle=0=\langle j,n|P_{\mathrm{Baker}}|m,n\rangle. \tag{45}$$

Since the second and third factors in Eq. (44) are still present whenever $j+m$ is even, and correspondingly for the case when $k+n$ is even, we have even more vanishing matrix elements. The matrix elements of $P_{\mathrm{Baker}}$ vanish for all $j+m$ even and $k\neq n$ and for all $k+n$ even and $j\neq m$. *Put another way, the only nonvanishing, off-diagonal matrix elements of $P_{\mathrm{Baker}}$ are those with $j+m$ and $k+n$ both odd and with $j<m$ and $n<k$.* The index inequalities reflect the triangular nature of the matrices **L** and **R**.

Earlier work,[2] utilizing resolvent operator and subdynamics[13] techniques on the $\beta$-adic generalization of the Baker map, has produced elegant constructions of linear combinations of the $|m,n\rangle$ that constitute a Jordan basis. These linear combinations, denoted here by $|p,\nu\rangle_J$, form a Jordan basis[2] that exhibits the $p+1$-fold degeneracy of the diagonal elements given in Eq. (43). This Jordan basis contains sets of $p+1$ states, $|p,\nu\rangle_J$, satisfying the identities

$$P_{\mathrm{Baker}}|p,\nu\rangle_J=\frac{1}{2^p}\,|p,\nu\rangle_J+|p,\nu+1\rangle_J \quad \text{for } \nu=0,1,2,\ldots,p-1, \tag{46}$$

$$P_{\mathrm{Baker}}|p,p\rangle_J=\frac{1}{2^p}\,|p,p\rangle_J. \tag{47}$$

Related states exist for the Baker K operator as well.[2] They satisfy the adjoint identities

$${}_J\langle p,\nu|P_{\mathrm{Baker}}=\frac{1}{2^p}\,{}_J\langle p,\nu|+{}_J\langle p,\nu-1| \quad \text{for } \nu=1,2,\ldots,p, \tag{48}$$

$${}_J\langle p,0|P_{\mathrm{Baker}}=\frac{1}{2^p}\,{}_J\langle p,0|. \tag{49}$$

The Bernoulli map is not invertible because each point $x$ in [0, 1] has two possible antecedents, $x/2$ or $x/2+\frac{1}{2}$. In the Baker map, however, this ambiguity is removed by coupling of the $x$ behavior to the $y$ behavior [cf. Eq. (34)]. Thus, the Baker map is invertible. The Baker F–P operator, however, is not self-adjoint. From Eq. (35), and the appropriate analogue of Eq. (13), it is straightforward to deduce the corresponding Baker K operator which is given by

$$\begin{aligned}K_{\mathrm{Baker}}g(x,y)&=g\left(2x,\,\frac{y}{2}\right)\Theta\left(\frac{1}{2}-x\right)\\ &\quad+g\left(2x-1,\,\frac{y}{2}+\frac{1}{2}\right)\Theta\left(x-\frac{1}{2}\right).\end{aligned} \tag{50}$$

This is identical with $P_{\mathrm{Baker}}$ if we interchange $x$ and $y$. Let us define

$${}_J\{p,\nu|=|p,p-\nu\rangle_J \quad \text{with } x\leftrightarrow y. \tag{51}$$

This guarantees the ${}_J\{p,\nu|$'s satisfy Eqs. (48) and (49). Below, we will eventually convert the ${}_J\{p,\nu|$'s into ${}_J\langle p,\nu|$'s by a renormalization.

A remarkably straightforward construction of these right-sided and left-sided Jordan states for $P_{\mathrm{Baker}}$ is obtained as follows. The off-diagonal matrix elements given in Eq. (42) are nonzero only if $j+m$ and $k+n$ are both odd and $j<m$ and $n<k$. Therefore, it is obvious that

$$|p,p\rangle_J=|0,p\rangle, \tag{52}$$

$$_J\{p,0|=\langle p,0|, \tag{53}$$

since an application of $P_{\rm Baker}$ reproduces Eqs. (47) and (49). Notice that in this case

$$\begin{aligned}_J\{p,0|&=(|p,p\rangle_J \text{ with } x\leftrightarrow y)\\&=(|0,p\rangle \text{ with } x\leftrightarrow y)=R_0(y)L_p(x)=\langle p,0|,\end{aligned} \tag{54}$$

where we have used Eqs. (51), (52), (38), and (39). Thus $_J\{p,0|={}_J\langle p,0|$, but in general we will find that the $_J\{p,\nu|$'s given by Eq. (51) will require renormalization before they satisfy the biorthonormality requirement

$$_J\langle p,\nu|p',\nu'\rangle_J=\delta_{pp'}\delta_{\nu\nu'}. \tag{55}$$

The matrix element index inequalities given with Eq. (42) also suggest the expansion

$$|p,p-j\rangle_J=\alpha_p^{(j)}\Big(|j,p-1\rangle+\sum_{k=0}^{j-1}\sum_{m>p-j}|k,m\rangle C_m^{(k;j,p)}\quad \text{for } j=1,2,\ldots,p, \tag{56}$$

$C_m^{(k;j,p)}\neq 0$ only if $k<j$ and $m>p-j$. We repeatedly use this implicitly below. Substituting Eq. (56) into both sides of Eq. (46) and projecting onto $\langle k,m|$ yields

$$\begin{aligned}&\alpha_p^{(j)}\Big(\langle k,m|P_{\rm Baker}|j,p-j\rangle+\sum_{r=0}^{j-1}\sum_{s>p-j}\langle k,m|P_{\rm Baker}|r,s\rangle C_s^{(r;j,p)}\\&=\frac{1}{2^p}\,\alpha_p^{(j)}\left(\langle k,m|j,p-j\rangle+\sum_{r=0}^{j-1}\sum_{s>p-j}\langle k,m|r,s\rangle C_s^{(r;j,p)}\right)\\&\quad+\alpha_p^{(j-1)}\left(\langle k,m|j-1,p-(j-1)\rangle+\sum_{r=0}^{j-2}\sum_{s>p-(j-1)}\langle k,m|r,s\rangle C_s^{(r;j-1,p)}\right).\end{aligned} \tag{57}$$

Repeated application of the biorthonormality, Eq. (40), yields for $k$ in $[0,\,j-1]$ and $m>p-j$

$$\begin{aligned}&\frac{1}{2^p}\,(\delta_{kj}\delta_{m\ \ p-j}+C_m^{(k;j,p)})\\&=\langle k,m|P_{\rm Baker}|j,p-j\rangle+\sum_{k\leq r}^{j-1}\sum_{s>p-j}^{m}\langle k,m|P_{\rm Baker}|r,s\rangle\\&\quad\times C_s^{(r;j,p)}-\frac{\alpha_p^{(j-1)}}{\alpha_p^{(j)}}\,(\delta_{k\ \ j-1}\delta_{m\ \ p-(j-1)}+C_m^{(k;j-1,p)}).\end{aligned} \tag{58}$$

If $k=j$, the second term on the left-hand side and the second, third, and fourth terms on the right-hand side all vanish, leaving a special case of Eq. (43):

$$\frac{1}{2^p}=\langle j,p-j|P_{\rm Baker}|j,p-j\rangle. \tag{59}$$

Because of Eq. (45), either both $k=r$ and $m=s$ in the second term on the right-hand side of Eq. (58), or neither equality holds. Thus, we may separate the $k=r$ and $m=s$ term from the others, move it to the left-hand side, and obtain the fundamental recursion formula for $k<j$:

$$\begin{aligned}\left(\frac{1}{2^p}-\frac{1}{2^{m+k}}\right)C_m^{(k;j,p)}&=\langle k,m|P_{\rm Baker}|j,p-j\rangle\\&\quad+\sum_{k\leq r}^{j-1}\sum_{s>p-j}^{m}\langle k,m|P_{\rm Baker}|r,s\rangle\\&\quad\times C_s^{(r;j,p)}\\&\quad-\frac{\alpha_p^{(j-1)}}{\alpha_p^{(j)}}\,(\delta_{k\ \ j-1}\delta_{m\ \ p-(j-1)}\\&\quad+C_m^{(k;j-1,p)}).\end{aligned} \tag{60}$$

The recursive iteration of these equations proceeds in the order: $j=0,1,2,\ldots,$ and $k=j-1,j-2,\ldots,2,1,0.$

The special case $k=j-1$ and $m=p-(j-1)$ yields

$$\langle j-1,p-(j-1)|P_{\rm Baker}|j,p-j\rangle=\frac{\alpha_p^{(j-1)}}{\alpha_p^{(j)}}. \tag{61}$$

Since $|p,p\rangle_J=|0,p\rangle$, we set $\alpha_p^{(0)}=1$ in the recursion for the $\alpha_p^{(j)}$'s, Eq. (61).

In order to fully grasp the nature of this recursion method, it is extremely illuminating to work out a few cases in detail. As will be seen, $p=1$ and $p=2$ are straightforward, but $p=3$ already introduces important subtleties. In discussing $p=3$, we will prove several identities applicable to the arbitrary $p$ case.

For $p=1$, $|1,1\rangle_J=|0,1\rangle$. Thus, only $j=1$ is allowed. Therefore, $k=0$. Equation (60) reduces to

$$\left(\frac{1}{2}-\frac{1}{2^m}\right)C_m^{(0;1,1)}=\langle 0,m|P_{\rm Baker}|1,0\rangle-\frac{\alpha_1^{(0)}}{\alpha_1^{(1)}}\,\delta_{m1}. \tag{62}$$

For $m>1$,

$$C_m^{(0;1,1)}=\frac{1}{\frac{1}{2}-\frac{1}{2^m}}\,\langle 0,m|P_{\rm Baker}|1,0\rangle. \tag{63}$$

$C_1^{(0;1,1)}$ is undetermined since $m=1$ implies that Eq. (62) yields

$$\alpha_1^{(1)}=\frac{1}{\langle 0,1|P_{\rm Baker}|1,0\rangle}. \tag{64}$$

Therefore, Eq. (56) implies

$$|1,0\rangle_J=\alpha_1^{(1)}\left(|1,0\rangle+\sum_{m>0}|0,m\rangle C_m^{(0;1,1)}\right). \tag{65}$$

From Eq. (51), we have

$$_J\{1,0| = (|1,1\rangle_J \text{ with } x\leftrightarrow y) = (|0,1\rangle \text{ with } x\leftrightarrow y) = \langle 1,0|. \tag{66}$$

Therefore,

$$_J\{1,0|1,0\rangle_J = \alpha_1^{(1)} \tag{67}$$

and

$$_J\langle 1,0| = \frac{1}{\alpha_1^{(1)}}\ {}_J\{1,0| = \frac{1}{\alpha_1^{(1)}}\langle 1,0|. \tag{68}$$

provides the renormalization needed in Eq. (55). Similarly,

$$_J\{1,1| = (|1,0\rangle_J \text{ with } x\leftrightarrow y) = \alpha_1^{(1)}\left(\langle 0,1| + \sum_{m>0} C_m^{(0;1,1)}\langle m,0|\right). \tag{69}$$

Therefore,

$$_J\{1,1|1,1\rangle_J = \alpha_1^{(1)} \tag{70}$$

and

$$_J\langle 1,1| = \frac{1}{\alpha_1^{(1)}}\ {}_J\{1,1| = \langle 0,1| + \sum_{m>0} C_m^{(0;1,1)}\langle m,0|. \tag{71}$$

Finally, we must have, according to Eq. (55),

$$_J\langle 1,1|1,0\rangle_J = {}_J\langle 1,0|1,1\rangle_J = 0. \tag{72}$$

The second inner product is automatic, but the first requires

$$2\alpha_1^{(1)} C_1^{(0;1,1)} = 0. \tag{73}$$

Thus, $C_1^{(0;1,1)}$ is finally determined; it vanishes.

For $p=2, |2,2\rangle_J = |0,2\rangle$ and $_J\{2,0| = \langle 2,0|$. The cases are $j=1$ and $k=0$, or $j=2$ and $k=1$, or 0.

For $j=1$ and $k=0$, we obtain

$$\left(\frac{1}{2^2} - \frac{1}{2^m}\right) C_m^{(0;1,2)} = \langle 0,m|P_{\text{Baker}}|1,1\rangle - \frac{\alpha_2^{(0)}}{\alpha_2^{(1)}}\delta_{m2}. \tag{74}$$

For $m>2$,

$$C_m^{(0;1,2)} = \frac{1}{\frac{1}{2^2} - \frac{1}{2^m}}\langle 0,m|P_{\text{Baker}}|1,1\rangle. \tag{75}$$

$C_2^{(0;1,2)}$ is undetermined since $m=2$ implies

$$\alpha_2^{(1)} = \frac{1}{\langle 0,2|P_{\text{Baker}}|1,1\rangle}. \tag{76}$$

Therefore,

$$|2,1\rangle_J = \alpha_2^{(1)}\left(|1,1\rangle + \sum_{m>1} C_m^{(0;1,2)}|0,m\rangle\right) \tag{77}$$

and

$$_J\{2,1| = (|2,1\rangle_J \text{ with } x\leftrightarrow y) = \alpha_2^{(1)}\left(\langle 1,1| + \sum_{m>1} C_m^{(0;1,2)}\langle m,0|\right). \tag{78}$$

Since

$$_J\{2,1|2,1\rangle_J = (\alpha_2^{(1)})^2, \tag{79}$$

then

$$_J\langle 2,1| = \frac{1}{(\alpha_2^{(1)})^2}\ {}_J\{2,1| = \frac{1}{\alpha_2^{(1)}}\left(\langle 1,1| + \sum_{m>1} C_m^{(0;1,2)}\langle m,0|\right). \tag{80}$$

For $j=2$ and $k=1$, we obtain

$$\left(\frac{1}{2^2} - \frac{1}{2^{m+1}}\right) C_m^{(1;2,2)} = \langle 1,m|P_{\text{Baker}}|2,0\rangle - \frac{\alpha_2^{(1)}}{\alpha_2^{(2)}}\delta_{m1}. \tag{81}$$

For $m>1$,

$$C_m^{(1;2,2)} = \frac{1}{\frac{1}{2^2} - \frac{1}{2^{m+1}}}\langle 1,m|P_{\text{Baker}}|2,0\rangle. \tag{82}$$

$C_1^{(1;2,2)}$ is undetermined since $m=1$ implies

$$\alpha_2^{(2)} = \frac{\alpha_2^{(1)}}{\langle 1,1|P_{\text{Baker}}|2,0\rangle}. \tag{83}$$

For $j=2$ and $k=0$, we obtain

$$\left(\frac{1}{2^2} - \frac{1}{2^m}\right) C_m^{(0;2,2)} = \langle 0,m|P_{\text{Baker}}|2,0\rangle + \sum_{s>0}^{m-1}\langle 0,m|P_{\text{Baker}}|1,s\rangle C_s^{(1;2,2)} - \frac{\alpha_2^{(1)}}{\alpha_2^{(2)}} C_m^{(0;1,2)}. \tag{84}$$

Notice that the first term on the right-hand side vanishes because $0+2$ is even. For $m>2$,

$$C_m^{(0;2,2)} = \frac{1}{\frac{1}{2^2} - \frac{1}{2^m}}\left(\sum_{s>0}^{m-1}\langle 0,m|P_{\text{Baker}}|1,s\rangle C_s^{(1;2,2)} - \frac{\alpha_2^{(1)}}{\alpha_2^{(2)}} C_m^{(0;1,2)}\right). \tag{85}$$

$C_2^{(0;2,2)}$ is undetermined since $m=2$ implies

$$0 = \langle 0,2|P_{\text{Baker}}|1,1\rangle C_1^{(1;2,2)} - \frac{\alpha_2^{(1)}}{\alpha_2^{(2)}} C_2^{(0;1,2)}. \tag{86}$$

Therefore,

$$|2,0\rangle_J = \alpha_2^{(2)}\left(|2,0\rangle + \sum_{m>0} C_m^{(1;2,2)}|1,m\rangle + \sum_{m>1} C_m^{(0;2,2)}|0,m\rangle\right). \tag{87}$$

The absence of the $m=1$ term in the last expression on the right-hand side follows from Eq. (84) because the first term on the right-hand side of Eq. (84) vanishes, the second term's summation has no summand for $m=1$, and $C_1^{(0;1,2)}=0$ by the $m>p-j$ rule for the $C_m^{(k;j,p)}$'s. Thus $C_1^{(0;2,2)}=0$. Therefore,

$$ {}_J\{2,2| = (|2,0\rangle_J \quad \text{with} \quad x\leftrightarrow y) = \alpha_2^{(2)}\left(\langle 0,2| + \sum_{m>0} C_m^{(1;2,2)} \times \langle m,1| + \sum_{m>1} C_m^{(0;2,2)}\langle m,0|\right). \tag{88} $$

Since

$$ {}_J\{2,2|2,2\rangle_J = \alpha_2^{(2)}, \tag{89} $$

then

$$ {}_J\langle 2,2| = \frac{1}{\alpha_2^{(2)}}\, {}_J\{2,2| = \langle 0,2| + \sum_{m>0} C_m^{(1;2,2)}\langle m,1| + \sum_{m>1} C_m^{(0;2,2)}\langle m,0|. \tag{90} $$

From Eq. (87), it is clear that

$$ {}_J\{2,0|2,0\rangle_J = \alpha_2^{(2)}. \tag{91} $$

Therefore,

$$ {}_J\langle 2,0| = \frac{1}{\alpha_2^{(2)}}\langle 2,0|. \tag{92} $$

We are now in a position to check biorthonormality, Eq. (55). We obtain

$$ \begin{aligned} {}_J\langle 2,2|2,0\rangle_J &= \alpha_2^{(2)}(C_2^{(0;2,2)} + (C_1^{(1;2,2)})^2 + C_2^{(0;2,2)}),\\ {}_J\langle 2,2|2,1\rangle_J &= \alpha_2^{(1)}(C_2^{(0;1,2)} + C_1^{(1;2,2)}),\\ {}_J\langle 2,2|2,2\rangle_J &= 1,\\ {}_J\langle 2,1|2,0\rangle_J &= \frac{\alpha_2^{(2)}}{\alpha_2^{(1)}}(C_1^{(1;2,2)} + C_2^{(0;1,2)}),\\ {}_J\langle 2,1|2,1\rangle_J &= 1,\\ {}_J\langle 2,1|2,2\rangle_J &= 0,\\ {}_J\langle 2,0|2,0\rangle_J &= 1,\\ {}_J\langle 2,0|2,1\rangle_J &= 0,\\ {}_J\langle 2,0|2,2\rangle_J &= 0. \end{aligned} \tag{93} $$

These results are consistent with Eq. (55) if

$$ C_2^{(0;1,2)} = C_1^{(1;2,2)} = C_2^{(0;2,2)} = 0. \tag{94} $$

Once again, the undetermined expansion coefficients are finally determined; they vanish. Moreover, these results are consistent with Eq. (86).

For $p=3$, $|3,3\rangle_J = |0,3\rangle$ and ${}_J\{3,0| = \langle 3,0|$. The cases are $j=1$ and $k=0$, or $j=2$ and $k=1$ or 0, or $j=3$ and $k=2$ or 1 or 0, a total of six cases. Recall the conditions for the nonvanishing of the expansion coefficients given just below Eq. (56). Only results will be given since the method should now be clear.

$j=1,\ k=0{:}\ m>2,$

$$ C_m^{(0;1,3)} = \frac{1}{\frac{1}{2^3} - \frac{1}{2^m}}\langle 0,m|P_{\text{Baker}}|1,2\rangle \quad \text{for} \quad m>3. \tag{95} $$

$C_3^{(0;1,3)}$ is undetermined since $m=3$ implies

$$ \alpha_3^{(1)} = \frac{1}{\langle 0,3|P_{\text{Baker}}|1,2\rangle}. \tag{96} $$

The associated Jordan state candidates are

$$ |3,2\rangle_J = \alpha_3^{(1)}\left(|1,2\rangle + \sum_{m>3} C_m^{(0;1,3)}|0,m\rangle + C_3^{(0;1,3)}|0,3\rangle\right), \tag{97} $$

$$ {}_J\{3,1| = \alpha_3^{(1)}\left(\langle 2,1| + \sum_{m>3} C_m^{(0;1,3)}\langle m,0| + C_3^{(0;1,3)}\langle 3,0|\right). \tag{98} $$

$j=2,\ k=1{:}\ m>1,$

$$ C_m^{(1;2,3)} = \frac{1}{\frac{1}{2^3} - \frac{1}{2^{m+1}}}\langle 1,m|P_{\text{Baker}}|2,1\rangle \quad \text{for} \quad m>2. \tag{99} $$

$C_2^{(1;2,3)}$ is undetermined since $m=2$ implies

$$ \alpha_3^{(2)} = \frac{\alpha_3^{(1)}}{\langle 1,2|P_{\text{Baker}}|2,1\rangle}. \tag{100} $$

$j=2,\ k=0{:}\ m>1,$

$$ C_m^{(0;2,3)} = \frac{1}{\frac{1}{2^3} - \frac{1}{2^m}}\left(\sum_{s>1}^{m-1}\langle 0,m|P_{\text{Baker}}|1,s\rangle C_s^{(1;2,3)} - \frac{\alpha_3^{(1)}}{\alpha_3^{(2)}} C_m^{(0;1,3)}\right) \quad \text{for} \quad m>3. \tag{101} $$

$C_3^{(0;2,3)}$ is undetermined since $m=3$ implies

$$ 0 = \langle 0,3|P_{\text{Baker}}|1,2\rangle C_2^{(1;2,3)} - \frac{\alpha_3^{(1)}}{\alpha_3^{(2)}} C_3^{(0;1,3)}. \tag{102} $$

In this case, the possibility of $m=2$ is excluded because all terms on the right-hand side of Eq. (60) vanish for one reason or another [Eq. (101) must be used to see that $C_2^{(0;2,3)}=0$].

The associated Jordan state candidates are

$$ |3,1\rangle_J = \alpha_3^{(2)}\left(|2,1\rangle + \sum_{m>3} C_m^{(0;2,3)}|0,m\rangle + C_3^{(0;2,3)}|0,3\rangle + \sum_{m>2} C_m^{(1;2,3)}|1,m\rangle + C_2^{(1;2,3)}|1,2\rangle\right), \tag{103} $$

$$ {}_J\{3,2| = \alpha_3^{(2)}\left( \langle 1,2| + \sum_{m>3} C_m^{(0;2,3)}\langle m,0| + C_3^{(0;2,3)}\langle 3,0| + \sum_{m>2} C_m^{(1;2,3)}\langle m,1| + C_2^{(1;2,3)}\langle 2,1| \right). \tag{104} $$

$j=3,\ k=2: m>0,$

$$ C_m^{(2;3,3)} = \frac{1}{\frac{1}{2^3}-\frac{1}{2^{m+2}}}\langle 2,m|P_{\text{Baker}}|3,0\rangle \quad \text{for } m>1. \tag{105} $$

$C_1^{(2;3,3)}$ is undetermined since $m=1$ implies

$$ \alpha_3^{(3)} = \frac{\alpha_3^{(2)}}{\langle 2,1|P_{\text{Baker}}|3,0\rangle} \tag{106} $$

$j=3,\ k=1: m>0,$

$$ C_m^{(1;3,3)} = \frac{1}{\frac{1}{2^3}-\frac{1}{2^{m+1}}}\left( \sum_{s>0}^{m-1} \langle 1,m|P_{\text{Baker}}|2,s\rangle C_s^{(2;3,3)} - \frac{\alpha_3^{(2)}}{\alpha_3^{(3)}} C_m^{(1;2,3)} \right) \quad \text{for } m>2, \tag{107} $$

$C_2^{(1;3,3)}$ is undetermined since $m=2$ implies

$$ 0 = \langle 1,2|P_{\text{Baker}}|2,1\rangle C_1^{(2;3,3)} - \frac{\alpha_3^{(2)}}{\alpha_3^{(3)}} C_2^{(1;2,3)}. \tag{108} $$

In this case, the possibility of $m=1$ is excluded because all terms on the right-hand side of Eq. (60) vanish for one reason or another.

$j=3,\ k=0:\ m>0,$

$$ C_m^{(0;3,3)} = \frac{1}{\frac{1}{2^3}-\frac{1}{2^{m}}}\left( \langle 0,m|P_{\text{Baker}}|3,0\rangle + \sum_{s>0}^{m-1} \langle 0,m|P_{\text{Baker}}|1,s\rangle C_s^{(1;3,3)} - \frac{\alpha_3^{(2)}}{\alpha_3^{(3)}} C_m^{(0;2,3)} \right) \quad \text{for } m>3. \tag{109} $$

$C_3^{(0;3,3)}$ is undetermined since $m=3$ implies

$$ 0 = \langle 0,3|P_{\text{Baker}}|3,0\rangle + \langle 0,3|P_{\text{Baker}}|1,2\rangle C_2^{(1;3,3)} - \frac{\alpha_3^{(2)}}{\alpha_3^{(3)}} C_3^{(0;2,3)}. \tag{110} $$

In this case, the last case of the six cases, $m=2$ and $m=1$ are still possibilities. Equation (60) implies

$$ \left(\frac{1}{2^3}-\frac{1}{2^2}\right) C_2^{(0;3,3)} = \langle 0,2|P_{\text{Baker}}|1,1\rangle C_1^{(1;3,3)} - \frac{\alpha_3^{(2)}}{\alpha_3^{(3)}} C_2^{(0;2,3)}, \tag{111} $$

$$ \left(\frac{1}{2^3}-\frac{1}{2}\right) C_1^{(0;3,3)} = \langle 0,1|P_{\text{Baker}}|3,0\rangle. \tag{112} $$

The associated Jordan state candidates are

$$ |3,0\rangle_J = \alpha_3^{(3)}\left( |3,0\rangle + \sum_{m>3} C_m^{(0;3,3)}|0,m\rangle + C_3^{(0;3,3)}|0,3\rangle + C_2^{(0;3,3)}|0,2\rangle + C_1^{(0;3,3)}|0,1\rangle + \sum_{m>2} C_m^{(1;3,3)}|1,m\rangle + C_2^{(1;3,3)}|1,2\rangle + \sum_{m>1} C_m^{(2;3,3)}|2,m\rangle + C_1^{(2;3,3)}|2,1\rangle \right), \tag{113} $$

$$ {}_J\{3,3| = \alpha_3^{(3)}\left( \langle 0,3| + \sum_{m>3} C_m^{(0;3,3)}\langle m,0| + C_3^{(0;3,3)} \times\langle 3,0| + C_2^{(0;3,3)}\langle 2,0| + C_1^{(0;3,3)}\langle 1,0| + \sum_{m>2} C_m^{(1;3,3)}\langle m,1| + C_2^{(1;3,3)}\langle 2,1| + \sum_{m>1} C_m^{(2;3,3)}\langle m,2| + C_1^{(2;3,3)}\langle 1,2| \right). \tag{114} $$

For $p=3$, the undetermined coefficients are $C_3^{(0;1,3)}$, $C_2^{(1;2,3)}$, $C_3^{(0;2,3)}$, $C_1^{(2;3,3)}$, $C_2^{(1;3,3)}$, and $C_3^{(0;3,3)}$. Moreover Eqs. (102), (108), and (110) must be satisfied, and Eq. (110) makes it clear that all of these coefficients cannot vanish. Checking biorthonormality, Eq. (55), yields

$$ \begin{aligned} {}_J\{3,3|3,3\rangle_J &= \alpha_3^{(3)}, \\ {}_J\{3,2|3,2\rangle_J &= \alpha_3^{(2)}\alpha_3^{(1)}, \\ {}_J\{3,1|3,1\rangle_J &= \alpha_3^{(1)}\alpha_3^{(2)}, \\ {}_J\{3,0|3,0\rangle_J &= \alpha_3^{(3)}. \end{aligned} \tag{115} $$

Therefore,

$$ {}_J\langle 3,0| = \frac{1}{\alpha_3^{(3)}}\langle 3,0|, \tag{116} $$

$$ {}_J\langle 3,1| = \frac{1}{\alpha_3^{(2)}}\left( \langle 2,1| + \sum_{m>3} C_m^{(0;1,3)}\langle m,0| + C_3^{(0;1,3)}\langle 3,0| \right), \tag{117} $$

$$ {}_J\langle 3,2| = \frac{1}{\alpha_3^{(1)}}\left( \langle 1,2| + \sum_{m>3} C_m^{(0;2,3)}\langle m,0| + C_3^{(0;2,3)}\langle 3,0| + \sum_{m>2} C_m^{(1;2,3)}\langle m,1| + C_2^{(1;2,3)}\langle 2,1| \right), \tag{118} $$

$$
{}_J\langle 3,3| = \langle 0,3| + \sum_{m>3} C_m^{(0;3,3)} \langle m,0| + C_3^{(0;3,3)} \langle 3,0| + C_2^{(0;3,3)} \langle 2,0| + C_1^{(0;3,3)} \langle 1,0| + \sum_{m>2} C_m^{(1;3,3)} \langle m,1| + C_2^{(1;3,3)} \langle 2,1| + \sum_{m>1} C_m^{(2;3,3)} \times \langle m,2| + C_1^{(2;3,3)} \langle 1,2|. \tag{119}
$$

These are candidates, for the renormalized left-sided Jordan states. However, further checks of biorthonormality reveal that orthogonality is not always satisfied. The cases that are not identically zero are

$$
{}_J\langle 3,1|3,0\rangle_J = \frac{\alpha_3^{(3)}}{\alpha_3^{(2)}} \left(C_1^{(2;3,3)} + C_3^{(0;1,3)}\right), \tag{120}
$$

$$
{}_J\langle 3,2|3,0\rangle_J = \frac{\alpha_3^{(3)}}{\alpha_3^{(1)}} \left(C_2^{(1;3,3)} + C_3^{(0;2,3)} + C_2^{(1;2,3)} C_1^{(2;3,3)}\right), \tag{121}
$$

$$
{}_J\langle 3,2|3,1\rangle_J = \frac{\alpha_3^{(2)}}{\alpha_3^{(1)}} \left(2C_2^{(1;2,3)}\right), \tag{122}
$$

$$
{}_J\langle 3,3|3,0\rangle_J = \alpha_3^{(3)} \left(2C_3^{(0;3,3)} + (C_2^{(2;3,3)})^2 + 2C_2^{(1;3,3)} C_1^{(2;3,3)}\right), \tag{123}
$$

$$
{}_J\langle 3,3|3,1\rangle_J = \alpha_3^{(2)} \left(C_3^{(0;2,3)} + C_2^{(1;2,3)} C_1^{(2;3,3)} + C_2^{(1;3,3)}\right), \tag{124}
$$

$$
{}_J\langle 3,3|3,2\rangle_J = \alpha_3^{(1)} \left(C_3^{(0;1,3)} + C_1^{(2;3,3)}\right). \tag{125}
$$

This circumstance is remedied by application of the Gram–Schmidt orthogonalization procedure.[2c]

We get genuine Jordan states as follows. The right-sided states are already given by Eq. (56). It is the left-sided states that require further work. Recall that $\alpha_p^{(0)} = 1$, and write

$$
{}_J\langle p,0| = \frac{1}{\alpha_p^{(0)} \alpha_p^{(p)}} \; {}_J\{p,0| \tag{126}
$$

and for $r = 1,2,\ldots,p$

$$
{}_J\langle p,r| = \frac{1}{\alpha_p^{(r)} \alpha_p^{(p-r)}} \times \left( {}_J\{p,r| - \sum_{s=0}^{r-1} {}_J\{p,r|p,s\rangle_J \; {}_J\langle p,s| \right). \tag{127}
$$

To show that both Eqs. (48) and (49) and Eq. (55) are now satisfied, we need two lemmas.

*Lemma 1:* For $r$ in $[0, p-1]$ and $s$ in $[0, p-1]$,

$$
{}_J\{p,r|p,s\rangle_J = {}_J\{p,r+1|p,s+1\rangle_J . \tag{128}
$$

*Proof:* From Eqs. (46)–(49), it follows that

$$
\left( P_{\text{Baker}} - \frac{1}{2^p} \right) |p,s\rangle_J = |p,s+1\rangle_J \tag{129}
$$

and

$$
{}_J\{p,r+1| \left( P_{\text{Baker}} - \frac{1}{2^p} \right) = {}_J\{p,r|. \tag{130}
$$

Therefore,

$$
{}_J\{p,r|p,s\rangle_J = {}_J\{p,r+1| \left( P_{\text{Baker}} - \frac{1}{2^p} \right) |p,s\rangle_J = {}_J\{p,r+1|p,s+1\rangle_J \tag{131}
$$

*Lemma 2:* For $r = 1,2,\ldots,p$,

$$
\frac{\alpha_p^{(r-1)} \alpha_p^{(p-(r-1))}}{\alpha_p^{(r)} \alpha_p^{(p-r)}} = 1. \tag{132}
$$

*Proof:* From Eq. (61), it follows that

$$
\frac{\alpha_p^{(r-1)} \alpha_p^{(p-(r-1))}}{\alpha_p^{(r)} \alpha_p^{(p-r)}} = \frac{\langle r-1, p-(r-1) | P_{\text{Baker}} | r, p-r \rangle}{\langle p-r, r | P_{\text{Baker}} | p-(r-1), r-1 \rangle}. \tag{133}
$$

From Eq. (42), the statement of the lemma follows because of the $j \leftrightarrow n$ and $k \leftrightarrow m$ symmetry

$$
\langle j,k | P_{\text{Baker}} | m,n \rangle = \langle n,m | P_{\text{Baker}} | k,j \rangle. \tag{134}
$$

The proof of Eqs. (48) and (49) for the left-sided states given in Eqs. (126) and (127) uses the principle of induction. Recall that by construction, the states ${}_J\{p,r|$ satisfy Eqs. (48) and (49). Therefore, Eq. (126) implies that ${}_J\langle p,0|$ satisfies Eq. (49). Similarly, we verify Eq. (48) for ${}_J\langle p,1|$.

$$
\begin{aligned}
{}_J\langle p,1| P_{\text{Baker}} &= \frac{1}{\alpha_p^{(1)} \alpha_p^{(p-1)}} \left( {}_J\{p,1| P_{\text{Baker}} - {}_J\{p,1|p,0\rangle_J \; {}_J\langle p,0| P_{\text{Baker}} \right) \\
&= \frac{1}{\alpha_p^{(1)} \alpha_p^{(p-1)}} \left( \frac{1}{2^p} \; {}_J\{p,1| + {}_J\{p,0| - \frac{1}{2^p} \; {}_J\{p,1|p,0\rangle_J \; {}_J\langle p,0| \right) \\
&= \frac{1}{2^p} \; {}_J\langle p,1| + \frac{\alpha_p^{(0)} \alpha_p^{(p)}}{\alpha_p^{(1)} \alpha_p^{(p-1)}} \; {}_J\langle p,0| = \frac{1}{2^p} \; {}_J\langle p,1| + {}_J\langle p,0|,
\end{aligned} \tag{135}
$$

where the last equality follows from Lemma 2. Now assume that Eq. (48) has been verified for $\nu = 0,1,2,\ldots,r-1$. We show that it is true for $r$:

$$
\begin{aligned}
{}_J\langle p,r|P_{\text{Baker}} &= \frac{1}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\left({}_J\{p,r|P_{\text{Baker}}-\sum_{s=0}^{r-1}\ {}_J\{p,r|p,s\rangle_J\ {}_J\langle p,s|P_{\text{Baker}\ J}\right)\\
&= \frac{1}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\left(\frac{1}{2^p}\ {}_J\{p,r|+{}_J\{p,r-1|-\frac{1}{2^p}\sum_{s=0}^{r-1}\ {}_J\{p,r|p,s\rangle_J\ {}_J\langle p,s|-\sum_{s=1}^{r-1}\ {}_J\{p,r|p,s\rangle_J\ {}_J\langle p,s-1|\right)\\
&= \frac{1}{2^p}\ {}_J\langle p,r|+\frac{1}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\left({}_J\{p,r-1|-\sum_{s=1}^{r-1}\ {}_J\{p,r|p,s\rangle_J\ {}_J\langle p,s-1|\right)\\
&= \frac{1}{2^p}\ {}_J\langle p,r|+\frac{1}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\left({}_J\{p,r-1|-\sum_{s=1}^{r-1}\ {}_J\{p,r-1|p,s-1\rangle_J\ {}_J\langle p,s-1|\right)\\
&= \frac{1}{2^p}\ {}_J\langle p,r|+\frac{1}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\left({}_J\{p,r-1|-\sum_{s=0}^{r-2}\ {}_J\{p,r-1|p,s\rangle_J\ {}_J\langle p,s|\right)\\
&= \frac{1}{2^p}\ {}_J\langle p,r|+\frac{\alpha_p^{(r-1)}\alpha_p^{(p-(r-1))}}{\alpha_p^{(r)}\alpha_p^{(p-r)}}\ {}_J\langle p,r-1|=\frac{1}{2^p}\ {}_J\langle p,r|+{}_J\langle p,r-1|.
\end{aligned}
\tag{136}
$$

The fourth equality follows from Lemma 1 and the seventh equality follows from Lemma 2. Thus, by induction, Eqs. (48) and (49) have been verified for the left-sided Jordan state candidates of Eqs. (126) and (127). It remains to prove Eq. (55). From above, we now have the analogue of Eq. (130) which is

$$
{}_J\langle p,r+1|\left(P_{\text{Baker}}-\frac{1}{2^p}\right)={}_J\langle p,r|. \tag{137}
$$

Therefore, with Eq. (129) this implies for $r>0$

$$
\begin{aligned}
{}_J\langle p,r|p,r\rangle_J &= {}_J\langle p,r|\left(P_{\text{Baker}}-\frac{1}{2^p}\right)|p,r-1\rangle_J\\
&= {}_J\langle p,r-1|p,r-1\rangle_J.
\end{aligned}
\tag{138}
$$

Application of this result $r$ times yields

$$
{}_J\langle p,r|p,r\rangle_J={}_J\langle p,0|p,0\rangle_J \tag{139}
$$

Equations (56) and (126) imply

$$
{}_J\langle p,0|p,0\rangle_J=1. \tag{140}
$$

Thus, the diagonal part of Eq. (55) has been proved. Now consider an off-diagonal inner product with $r<s$,

$$
\begin{aligned}
{}_J\langle p,r|p,s\rangle_J &= {}_J\langle p,0|p,s-r\rangle_J\\
&= {}_J\langle p,0|\left(P_{\text{Baker}}-\frac{1}{2^p}\right)|p,s-r-1\rangle_J=0,
\end{aligned}
\tag{141}
$$

wherein the first equality follows from $r$ iterations of Eqs. (129) and (137) in succession, the second equality follows from Eq. (129), and the last equality is a consequence of Eq. (49). For $r>s$, an inductive argument is required. Clearly, for $r=1$, Eqs. (127) and (140) guarantee

$$
{}_J\langle p,1|p,0\rangle_J=0. \tag{142}
$$

For $r=2$, Eqs. (127) and (139)–(141) guarantee both

$$
{}_J\langle p,2|p,0\rangle_J=0, \tag{143}
$$

$$
{}_J\langle p,2|p,1\rangle_J=0. \tag{144}
$$

Therefore, assume that for some $r>0$ and for all $s<r$ we have shown that

$$
{}_J\langle p,r|p,s\rangle_J=0. \tag{145}
$$

We now show that for all $s<r+1$

$$
{}_J\langle p,r+1|p,s\rangle_J=0. \tag{146}
$$

*Proof:* From Eq. (127) we have

$$
\begin{aligned}
{}_J\langle p,r+1|p,s\rangle_J &= \frac{1}{\alpha_p^{(r+1)}\alpha_p^{(p-(r+1))}}\Bigg({}_J\{p,r+1|p,s\rangle_J\\
&\quad -\sum_{s'=0}^{r}\ {}_J\{p,r+1|p,s'\rangle_J\ {}_J\langle p,s'|p,s\rangle_J\Bigg)\\
&= \frac{1}{\alpha_p^{(r+1)}\alpha_p^{(p-(r+1))}}\Bigg({}_J\{p,r+1,|p,s\rangle_J\\
&\quad -\sum_{s'=0}^{r}\ {}_J\{p,r+1|p,s'\rangle_J\delta_{s's}\Bigg)\\
&= 0.
\end{aligned}
\tag{147}
$$

Thus, by induction, we have verified Eq. (145) in general for $s<r$. This completes the verification of Eq. (55).

It is now possible to give the spectral decomposition of the Baker F–P operator.[2c] We obtain

$$P_{\text{Baker}}=|0,0\rangle_J\ _J\langle 0,0|+\sum_{p=1}^{\infty}\left(\sum_{\nu=0}^{p}|p,\nu\rangle_J\frac{1}{2^p}\times{}_J\langle p,\nu|+\sum_{\nu=0}^{p-1}|p,\nu+1\rangle_J\ _J\langle p,\nu|\right). \tag{148}$$

Especially note the simple $\nu$-shift structure that embodies the off-diagonal terms in the Jordan canonical form[10] in the last term on the right-hand side.

## IV. RECURSION FORMULAS AND COMBINATORIAL IDENTITIES

Three different outcomes have occurred for the coefficients of eigenstates or Jordan states in terms of the expansion basis. In Eq. (12), the expansion coefficients for $R_m(x)$, the Bernoulli eigenfunctions, are given by a finite sum containing Bernoulli numbers, themselves generated recursively.[5,12] In Eq. (27), the expansion coefficients for $L_m(x)$, the Bernoulli–Koopman eigenstates, are given in closed form by an explicit function of the indices $m$ and $n$. Why isn't Eq. (12) of this form? Can its finite sum be executed in closed form for arbitrary $m$ and $n$? I don't think so. I think it is fully reduced as is. Equation (27), however, is also the solution to a fundamental recursion formula, Eq. (155), to be derived below. That the recursion formula, Eq. (155), has Eq. (27) as its solution can be explicitly verified for a finite number of special cases, but a direct, general proof that this is so for every choice of indices has not been found. The truth of Eq. (27) is proved by an independent argument (Appendix B) and not by solving the recursion. Finally, in Eq. (60), the expansion coefficients for $|p,\nu\rangle_J$, the Jordan basis, are given by a recursion formula. Is there a closed form solution to this recursion that gives an explicit formula for the $C_m^{(k;j,p)}$, in parallel with Eq. (27)? Perhaps the solution, if it exists, will be more like Eq. (12), i.e., requiring execution of finite sums for each choice of indices. Perhaps Eq. (60) is the optimal result.

The formula for the matrix elements, $L_{mn}$, Eq. (27), appears in Ref. 2a. It is a consequence of the subdynamics method[2,13] applied to the resolvent operator treatment of the F–P operator, [Eqs. (3.14) and (3.15) of Ref. 2a], all of which yields a recursion formula [Eq. (F1) of Ref. 2a], the solution to which is Eq. (27). The fundamental identity may be derived using Zwanzig's 1960 projection operator technique[14] and Laplace transforms. This approach requires several pages of proof, and the subsequent establishment of the recursion equation in Ref. 2a takes a few more. An explicit proof of Eq. (27) does not appear in Ref. 2a although the method to be shown here is implicit in Ref. 8. Below, we show how to arrive at the recursion formula directly from Eqs. (17) and (23), without use of the resolvent operator apparatus. This parallels the construction in the previous section in which a direct method produced the fundamental recursion formula, Eq. (60) [compare the subdynamics result, Eq. (14) of Ref. 2c].

Together, Eqs. (17) and (23) imply

$$K_B\sum_{k=m}^{\infty}L_{mk}\phi_k(x)=\frac{1}{2^m}\sum_{k=m}^{\infty}L_{mk}\phi_k(x). \tag{149}$$

Multiplication by $\phi_n(x)$ and integration over $x$ yields the matrix equation

$$L_{mn}=2^m\sum_{k=m}^{n}L_{mk}I_{kn} \tag{150}$$

in which $I_{kn}$ is defined by

$$I_{kn}=\int_0^1 dx\ \phi_n(x)K_B\phi_k(x)=\int_0^1 dx\ \phi_k(x)P_B\phi_n(x). \tag{151}$$

These matrix elements are straightforwardly evaluated using Eqs. (4) and (10)[2a] giving

$$I_{kn}=\frac{\sqrt{(2k+1)(2n+1)}}{2^k}\sum_{s=0}^{n-k}\left(-\frac{1}{2}\right)^s\times\frac{(k+n+s)!}{(n-k-s)!(2k+s+1)!s!}, \tag{152}$$

which is true only for $k\leqslant n$. Otherwise, it is zero. Clearly,

$$I_{nn}=\frac{1}{2^n}. \tag{153}$$

Thus, Eq. (150) may be rewritten as

$$L_{mn}=\left(\frac{2^m}{1-\frac{1}{2^{n-m}}}\right)\sum_{k=m}^{n-1}L_{mk}I_{kn}. \tag{154}$$

Since it is known that $L_{mm}=1$ [see Eq. (27)], we quickly obtain the fundamental recursion formula, Eq. (F1) of Ref. 2a.

$$L_{mn}=\left(\frac{2^m}{1-\frac{1}{2^{n-m}}}\right)\left(I_{mn}+\sum_{k=m+1}^{n-1}L_{mk}I_{kn}\right). \tag{155}$$

We have directly checked this recursion against Eq. (27) given Eq. (152). In fact, because these equations were established in Ref. 2a, originally we tried to prove Eq. (27) from this recursion and Eq. (152). Later, we discovered the nonrecursive proof of Eq. (27) given in Appendix B. It is, nevertheless, very striking to explicitly contemplate the simultaneous truth of Eqs. (27) and (155) (rewritten with $n=m+2k$):

$$\frac{(2m+2k-1)!}{(2k+1)!}=\sum_{j=0}^{k}\sum_{s=0}^{2k-2j}\left(-\frac{1}{2}\right)^s\frac{1}{2^{2j}}(2m+4j+1)\times\frac{(2m+2j+2k+s)!}{s!(2k-2j-s)!(2m+4j+s+1)!}\times\frac{(2m+2j-1)!}{(2j+1)!}. \tag{156}$$

That this combinatorial identity is true for any $m\geqslant 1$ and any $k\geqslant 0$ is amazing, as can be seen by checking it.

Also, before we discovered the proof of Eq. (27) given in Appendix B, we observed that Eq. (26) implies that there is an explicit finite summation yielding the matrix elements $L_{mn}$. From Eq. (12), we see that

$$R_{mm}=1. \tag{157}$$

for every $m\geqslant 1$. Therefore, Eq. (26) may be rewritten as

$$\mathbf{L}=(\mathbf{R}^{\dagger})^{-1}=(\mathbf{1}+\mathbf{T})^{-1}, \tag{158}$$

in which $\mathbf{1}$ is the identity and $\mathbf{T}$ is the off-diagonal part of $\mathbf{R}^{\dagger}$. Because $\mathbf{R}$ is lower triangular, $\mathbf{T}$ is strictly super-diagonal, and its successive powers are more and more super-diagonal. Thus the matrix element of $\mathbf{L}$ is given by a finite series [rather than by the infinite series seemingly implicit in the last expression on the right-hand side of Eq. (158)]

$$L_{m\ \ m+2k}=\delta_{k0}+(1-\delta_{k0})\sum_{j=1}^{k}((-T)^{j})_{m\ \ m+2k}. \tag{159}$$

Because of Eq. (12), we know $\mathbf{T}$ explicitly, and all that remains is to execute the summation in Eq. (159). Numerous checks for special values of $k$ have verified the consistency of Eqs. (159) and (27), a combinatorial identity involving the Bernoulli numbers.

## V. THE ENTROPY EVOLUTION EQUATION

On the one hand, it is known[4] that the Baker F–P operator, $P_{\text{Baker}}$, preserves Gibbs entropy, $S$, a functional of $f(x,y)$ defined by

$$S[f]=-\int_0^1 dx\int_0^1 dy\ f(x,y)\ln(f(x,y)). \tag{160}$$

That is,

$$S[P_{\text{Baker}}f]=S[f]. \tag{161}$$

In this case, the invariant, maximal entropy, equilibrium density under $P_{\text{Baker}}$ is the constant 1 over the entire unit square. Thus, the conditional entropy[4] and the entropy are identical. The proof of Eq. (161) is straightforward[4] and appears in Appendix D. On the other hand, if we contemplate an expansion for $f(x,y)$ in terms of the $|p,\nu\rangle_J$ basis for $P_{\text{Baker}}$, then we have the formal expansion

$$f(x,y)=\sum_{p=0}^{\infty}\sum_{\nu=0}^{p}c_{p\nu}|p,\nu\rangle_J, \tag{162}$$

and application of $P_{\text{Baker}}$ produces

$$P_{\text{Baker}}f(x,y)=\sum_{p=0}^{\infty}\sum_{\nu=0}^{p}c_{p\nu}\frac{1}{2^p}|p,\nu\rangle_J+\sum_{p=0}^{\infty}\sum_{\nu=0}^{p-1}c_{p\nu}|p,\nu+1\rangle_J. \tag{163}$$

The new states with argument $\nu+1$ will receive reciprocal powers of two prefactors upon successive application of $P_{\text{Baker}}$ as well as shifts to larger $\nu$-arguments, at least until $\nu=p$ is reached. These features are captured by the identity [cf. Eq. (58) of Ref. 2c] for the $n$-fold iteration of $P_{\text{Baker}}$ acting on $|p,\nu\rangle_J$ with $n>p-\nu$:

$$P_{\text{Baker}}^n|p,\nu\rangle_J=\sum_{k=0}^{p-\nu}\frac{n!}{k!(n-k)!}\frac{1}{2^{(n-k)p}}|p,\nu+k\rangle_J \tag{164}$$

in which the upper limit of the summation is $n$ for $n<p-\nu$. Apparently, successive applications of $P_{\text{Baker}}$ on $f(x,y)$ cause all but the invariant, equilibrium state, i.e., $|0,0\rangle_J$, which is constant and has eigenvalue 1, to decay (this argument is not uniform in $p$). The asymptotic result appears to be the equilibrium density which has maximal entropy. It would appear that the entropy has increased rather than having remained constant. On the other hand, the decays in Eq. (164) manifest the intrinsic irreversibility of the dynamics, as would be seen in correlation functions.[1d,2c]

That there is really no paradox here as a result of the inconsistency of Eqs. (161) and (164) is a consequence of the invalidity of the expansion for a probability distribution given by Eq. (162). This is due to the Schwartzian nature of the Jordan states. Thus the two sides of the equation are functions of very different character. This sort of subtlety in this kind of analysis has been discussed in Refs. 1d and 2c.

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## APPENDIX A: PROOF OF EQ. (12)

Equations (8) and (11) imply

$$R_{mn}=\int_0^1 dx\ R_m(x)\phi_n(x). \tag{A1}$$

A Bernoulli polynomial property[2,5,12] implies (note the $B$'s not $R$'s)

$$\frac{d^n}{dx^n}B_m(x)=\frac{m!}{(m-n)!}B_{m-n}(x). \tag{A2}$$

Together with Eq. (10), Eq. (A2) converts Eq. (A1) into

$$\begin{aligned}R_{mn}&=\frac{\sqrt{(2n+1)}}{n!}\int_0^1 dx\ R_m(x)\frac{d^n}{dx^n}(x(1-x))^n\\&=\frac{\sqrt{(2n+1)(2m+1)}}{n!}(-1)^{m+n}\frac{(2m)!m!}{(m!)^2(m-n)!}\\&\quad\times\int_0^1 dx\ B_{m-n}(x)(x(1-x))^n,\end{aligned} \tag{A3}$$

wherein an $n$-fold integration by parts has been used along with the fact that all boundary terms vanish. Clearly, for $n > m$, the result is 0. Moreover, since[5,12]

$$B_m(1-x)=(-1)^m B_m(x), \tag{A4}$$

these integrals vanish unless $m+n$ is even, as is proved by making the substitution $x \leftrightarrow 1-x$.

To finish the proof, we need explicit expressions for the integrals. These are most easily obtained from the generating function for Bernoulli polynomials:[2,5,12]

$$\frac{t\,\exp[xt]}{\exp[t]-1}=\sum_{m=0}^{\infty} B_m(x)\,\frac{t^m}{m!}. \tag{A5}$$

This implies that

$$\int_0^1 dx\,\frac{t\,\exp[xt]}{\exp[t]-1}\,(x(1-x))^n = \sum_{m=0}^{\infty} \int_0^1 dx\, B_m(x)(x(1-x))^n\,\frac{t^m}{m!}. \tag{A6}$$

The left-hand side may be rewritten as

$$\int_0^1 dx\,\frac{t\,\exp[xt]}{\exp[t]-1}\,(x(1-x))^n = \int_0^1 dx\,\frac{t}{\exp[t]-1}\sum_{q=0}^{\infty}\frac{t^q}{q!}\,x^{q+n}(1-x)^n. \tag{A7}$$

Using integration by parts $j$ times, one easily gets[2a]

$$\int_0^1 dx\, x^i(1-x)^j=\frac{j!\,i!}{(i+j+1)!}. \tag{A8}$$

Putting this into Eq. (A7) yields

$$\int_0^1 dx\,\frac{t\,\exp[xt]}{\exp[t]-1}\,(x(1-x))^n = \frac{t}{\exp[t]-1}\sum_{q=0}^{\infty}\frac{n!\,(q+n)!}{(2n+q+1)!\,q!}\,t^q. \tag{A9}$$

We now have an explicit expression for the left-hand side of Eq. (A6). Since the right-hand side of Eq. (A6) is a power series in $t$, we can obtain the $m$th term by taking $m$ $t$-derivatives, followed by setting $t=0$. This yields the identity

$$\int_0^1 dx\, B_m(x)(x(1-x))^n = \frac{d^m}{dt^m}\left(\frac{t}{\exp[t]-1}\sum_{q=0}^{\infty}\frac{n!\,(q+n)!}{(2n+q+1)!\,q!}\,t^q\right), \tag{A10}$$

wherein $t$ is set to 0 on the right-hand side after the differentiations. From Eq. (A5), we see that

$$\frac{t}{\exp[t]-1}=\sum_{m=0}^{\infty} B_m(0)\,\frac{t^m}{m!}=\sum_{m=0}^{\infty} B_m\,\frac{t^m}{m!}, \tag{A11}$$

wherein the second equality follows from the fact that the values of the Bernoulli polynomials at $x=0$ are just the Bernoulli numbers.[5] When we substitute this expression into Eq. (A10) and use the Leibnitz rule for the differentiations, followed by setting $t=0$, we finally get

$$\int_0^1 dx\, B_m(x)(x(1-x))^n = \sum_{r=0}^{m}\frac{m!}{r!\,(m-r)!}\,B_{m-r}\,\frac{n!\,(n+r)!}{(2n+r+1)!}. \tag{A12}$$

Looking back at Eq. (A3), we see that we need Eq. (A12) for the special case: $m$ replaced by $m-n$. The result is Eq. (12).

## APPENDIX B: PROOF OF EQ. (27)

In **I**, we proved that there is a class of functions that can be expanded in terms of the Bernoulli polynomials. Such expansions take the form

$$f(x)=\sum_{m=0}^{\infty} c_m B_m(x) \tag{B1}$$

with the coefficients given by the formula[5,12]

$$c_m=\frac{1}{m!}\,(f^{(m-1)}(1)-f^{(m-1)}(0)) \tag{B2}$$

where $f^{(k)}$ denotes the $k$th derivative of $f$ for $k\geqslant 1$ and

$$c_0=\int_0^1 dx\, f(x). \tag{B3}$$

A combination of Eqs. (24) and (25) and Eq. (6) implies

$$\phi_n(x)=\sum_{m=0}^{n} L_{mn}(-1)^m\sqrt{(2m+1)}\,\frac{(2m)!}{(m!)^2}\,B_m(x). \tag{B4}$$

Equation (B2) implies that

$$L_{mn}=(-1)^m\,\frac{1}{\sqrt{(2m+)}}\,\frac{(m!)^2}{(2m)!\,m!} \times(\phi_n^{(m-1)}(1)-\phi_n^{(m-1)}(0)). \tag{B5}$$

From Eq. (10), we see that

$$\phi_n^{(m-1)}(x)=\frac{\sqrt{(2n+1)}}{n!}\,\frac{d^{n+m-1}}{dx^{n+m-1}}\,(x(1-x))^n. \tag{B6}$$

An application of Leibnitz's rule generates

$$\phi_n^{(m-1)}(x)=\frac{\sqrt{(2n+1)}}{n!}\sum_{q=m-1}^{n+m-1}\frac{(n+m-1)!}{q!\,(n+m-1-q)!} \times\frac{n!}{(n-q)!}\,(-1)^{n+m+1-q}\,\frac{n!}{(q+1-m)!} \times x^{n-q}(1-x)^{q+1-m}. \tag{B7}$$

At $x=1$, only the $q=m-1$ term is nonzero and we get

$$\phi_n^{(m-1)}(1)=(-1)^n\sqrt{(2n+1)}\,\frac{(n+m-1)!}{(m-1)!(n-m+1)!}, \tag{B8}$$

while at $x=0$, only the $q=n$ term is nonzero and we get

$$\phi_n^{(m-1)}(0)=(-1)^{m+1}\sqrt{(2n+1)}\,\frac{(n+m-1)!}{(m-1)!(n-m+1)!}. \tag{B9}$$

Therefore,

$$\phi_n^{(m-1)}(1)-\phi_n^{(m-1)}(0)$$
$$=(-1)^n\sqrt{(2n+1)}\,\frac{(n+m-1)!}{(m-1)!(n-m+1)!}\times(1-(-1)^{n+m+1}). \tag{B10}$$

Putting this into Eq. (B5) produces Eq. (27) when $m+n$ is even, and zero otherwise.

## APPENDIX C: PROOF OF EQ. (42)

The left-hand side of Eq. (42) is expressible as

$$\begin{aligned}\langle j,k|P_{\text{Baker}}|m,n\rangle &=\int_0^1 dx\int_0^1 dy\ L_j(x)R_k(y)P_{\text{Baker}}R_m(x)L_n(y)\\ &=\int_0^1 dx\int_0^1 dy\ L_j(x)R_k(y)[R_m(x/2)L_n(2y)\Theta(1/2-y)\\ &\quad+R_m(x/2+1/2)L_n(2y-1)\Theta(y-1/2)]\\ &=\int_0^1 dx\ L_j(x)R_m(x/2)\int_0^{1/2}dy\ R_k(y)L_n(2y)\\ &\quad+\int_0^1 dx\ L_j(x)R_m(x/2+1/2)\int_{1/2}^1 dy\ R_k(y)L_n(2y-1)\\ &=\int_0^1 dx\ L_j(x)R_m(x/2)\,\frac{1}{2}\int_0^1 dy\ R_k(y/2)L_n(y)\\ &\quad+\int_0^1 dx\ L_j(x)R_m(x/2+1/2)\\ &\quad\times\frac{1}{2}\int_0^1 dy\ R_k(y/2+1/2)L_n(y).\end{aligned} \tag{C1}$$

Equation (A4) implies

$$R_m(1-x)=(-1)^m R_m(x). \tag{C2}$$

In parallel, Eq. (10) implies

$$\phi_n(1-x)=(-1)^n\phi_n(x), \tag{C3}$$

which with Eq. (23) implies

$$L_m(1-x)=(-1)^m L_m(x). \tag{C4}$$

Therefore,

$$\begin{aligned}\int_0^1 dx\ L_j(x)R_m(x/2+1/2) &=(-1)^m\int_0^1 dx\ L_j(x)R_m(1/2-x/2)\\ &=(-1)^m\int_0^1 dx\ L_j(1-x)R_m(x/2)\\ &=(-1)^{m+j}\int_0^1 dx\ L_j(x)R_m(x/2),\end{aligned} \tag{C5}$$

wherein the second equality utilized the substitution $x\rightarrow 1-x$, the first used Eq. (C2) and the third used Eq. (C4). A similar sequence of arguments enables us to also prove

$$\begin{aligned}\int_0^1 dy\ R_k(y/2+1/2)L_n(y) &=(-1)^k\int_0^1 dy\ R_k(1/2-y/2)L_n(y)\\ &=(-1)^k\int_0^1 dy\ R_k(y/2)L_n(1-y)\\ &=(-1)^{k+n}\int_0^1 dy\ R_k(y/2)L_n(y).\end{aligned} \tag{C6}$$

Putting Eqs. (C5) and (C6) into Eq. (C1) yields

$$\langle j,k|P_{\text{Baker}}|m,n\rangle=\int_0^1 dx\ L_j(x)R_m(x/2)\int_0^1 dy\ R_k(y/2)\times L_n(y)\,\frac{1}{2}\,[1+(-1)^{m+j+k+n}]. \tag{C7}$$

The remaining two integrals are of the same structure. Consider any function, $f(x)$, expandable in terms of the Bernoulli polynomials as in Eq. (B1). Using Eq. (6), this becomes

$$f(x)=\sum_m c_m(-1)^m \frac{1}{\sqrt{(2m+1)}}\frac{(m!)^2}{(2m)!}R_m(x). \tag{C8}$$

The biorthonormality expressed by Eq. (18) implies

$$\begin{aligned}\int_0^1 dx\, L_j(x)f(x)&=c_j(-1)^j \frac{1}{\sqrt{(2j+1)}}\frac{(j!)^2}{(2j)!}\\ &=(-1)^j \frac{1}{\sqrt{(2j+1)}}\frac{j!}{(2j)!}\\ &\quad\times(f^{(j-1)}(1)-f^{(j-1)}(0))\end{aligned} \tag{C9}$$

in which the second equality follows from Eq. (B2). This result and Eq. (C7) imply that we need the identity

$$\begin{aligned}R_m^{(j-1)}(x/2)&=(-1)^m\sqrt{(2m+1)}\,\frac{(2m)!}{(m!)^2}B_m^{(j-1)}(x/2)\\ &=(-1)^m\sqrt{(2m+1)}\,\frac{(2m)!}{(m!)^2}\\ &\quad\times\frac{m!}{(m-j+1)!}\frac{1}{2^{j-1}}B_{m-j+1}(x/2),\end{aligned} \tag{C10}$$

where the second equality follows from Eq. (A2). Therefore,

$$\begin{aligned}\int_0^1 dx\, L_j(x)R_m(x/2)&=(-1)^{j+m}\frac{\sqrt{(2m+1)}}{\sqrt{(2j+1)}}\frac{j!}{(2j)!}\\ &\quad\times\frac{(2m)!}{m!(m-j+1)!}\frac{1}{2^{j-1}}\\ &\quad\times(B_{m-j+1}(1/2)-B_{m-j+1}(0)).\end{aligned} \tag{C11}$$

Clearly, for $j\geqslant m+1$, this result vanishes. Thus $j\leqslant m$ is established. Treating the second integral in Eq. (C7) the same way completes the proof.

## APPENDIX D: PROOF OF EQ. (161)

Equation (35) implies

$$\begin{aligned}S[P_{\text{Baker}}f]&=-\int_0^1 dx\int_0^{1/2}dy\; f(x/2,2y)\ln(f(x/2,2y))\\ &\quad-\int_0^1 dx\int_{1/2}^1 dy\; f(x/2+1/2,2y-1)\\ &\quad\times\ln(f(x/2+1/2,2y-1))\\ &=-\int_0^1 dx\int_0^1 dy\Big[\frac{1}{2}f(x/2,y)\ln(f(x/2,y))\\ &\quad+\frac{1}{2}f(x/2+1/2,y)\ln(f(x/2+1/2,y))\Big]\\ &=-\int_0^{1/2}dx\int_0^1 dy\; 2\,\frac{1}{2}f(x,y)\ln(f(x,y))\\ &\quad-\int_{1/2}^1 dx\int_0^1 dy\; 2\,\frac{1}{2}f(x,y)\ln(f(x,y))\\ &=S[f],\end{aligned} \tag{D1}$$

wherein the second equality follows from variable changes $2y\rightarrow y$ and $2y-1\rightarrow y$, and the third equality follows from variable changes $x/2\rightarrow x$ and $x/2+\frac{1}{2}\rightarrow x$.

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