A Conjecture Regarding the Riemann Hypothesis

by

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Abstract

Numerical display of the behavior of strings originating inside the critical strip for the Dirichlet Eta function provides strong visual evidence for why the Riemann hypothesis is most likely true. A modified version of the reflection principle is used to justify this claim. The strings are generated by the action of the Dirichlet eta function on the unit interval of sigma's for each fixed value of t. The t strings can exhibit at least three types of behavior. They cannot intersect the origin, they can intersect the origin at a point emanating from sigma equal 0.5, or they can intersect themselves. If only the first two possibilities occur the Riemann hypothesis is true. If the third possibility occurs and the intersection point is also the origin then the Riemann hypothesis is false. Heuristic numerical evidence is presented that suggests the Riemann hypothesis is true. The results presented in this paper were made possible using *Wolfram Mathematica* 12.

Introduction

The Riemann hypothesis remains one of the outstanding problems in analytic number theory after almost 160 years [1]. It is concerned with the distribution of prime numbers and is encapsulated by the Riemann zeta function and an equivalent expression written exclusively in terms of primes found by Euler [2] much earlier (for real s).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1 - p^{-s}}$$

in which $s = \sigma + ti$, a complex number with real part σ and imaginary part coefficient t. The expressions above are valid for $\sigma > 1$ and are absolutely convergent there. In [1] Riemann used analytic continuation to extend the zeta function into the regime $0 < \sigma < 1$ (and into the regime $\sigma < 0$) and found the Dirichlet eta function equation (conjectured by Euler in 1749)

$$\eta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s})\zeta(s)$$

This result involves conditionally convergent series and must be handled with care. In the form

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^s} = \frac{1}{(1 - 2^{1-s})} \eta(s)$$

it provides a way of computing the zeta function for $0 < \sigma < 1$ in terms of the alternating eta series. The eta series is the difference between even n terms and odd n terms. If these terms are summed separately, the sub-series of all odd n and the sub-series of all even n diverge.

Also in [1] Riemann established the reflection formula for $0 < \sigma < 1$

$$\zeta(s) = 2^s \pi^{s-1} sin\left(\frac{\pi s}{2}\right) \Gamma(1 - s)\zeta(1 - s)$$

Using this equation together with the equation just above it yields the *reflection formula for the eta function*

$$\eta(s) = 2^s\pi^{s-1}sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\frac{(1-2^{1-s})}{(1-2^s)}\eta(1-s)$$

There are zeroes of zeta and eta of three kinds. There are some so-called trivial zeroes for $s = \sigma = -2, -4, \ldots$ created by the sine term. There are also trivial zeroes associated with the factor $\frac{(1-2^{1-s})}{(1-2^s)}$. These are of the form $1 - i\frac{k2\pi}{\ln(2)}$ for integer k. The remaining non-trivial zeroes of zeta are also non-trivial zeroes for eta and *visa versa*. The critical strip defined by $0 < \sigma < 1$ and $-\infty < t < \infty$ contains these non-trivial zeroes. The Riemann hypothesis is: $\boldsymbol{\sigma = 0.5}$ **for all non-trivial zeroes.** In 2004, Gourdon [3] verified the hypothesis for the first 10^{13} zeroes using an algorithm invented by Odlyzko [4].

A modified reflection formula

The reflection formulas for zeta and eta given in the introduction are most useful when considering zeroes of zeta/eta. The trivial zeroes account for all zeroes due to factors other than zeta/eta in the reflection equations. Only the non-trivial zeroes remain to be studied strictly inside the critical strip. The other factors do not vanish inside the critical strip. Here we will focus attention directly on the eta function. Two results will be of use later in the paper. The first result is

Lemma 1: if s_0 is a zero of eta then so is s_0*.

For general analytic functions, F, it is very easy to construct a counter-example to the statement that if z is a zero of F, then so is z*. The structure of eta, however, makes the statement of Lemma 1 true.

$$\eta(s_0) = 0 \rightarrow \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{s_0}} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\sigma_0}}Exp[-it_0ln(n)]$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\sigma_0}}(cos[t_0ln(n)]) - i\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^{\sigma_0}}sin[t_0ln(n)] = 0$$

Since both the real and the imaginary parts must vanish separately, one is free to replace the $-i$ by $+i$ in the last line and that implies $\eta(s_0 *) = 0$.

The second result is the *modified reflection formula* [5]

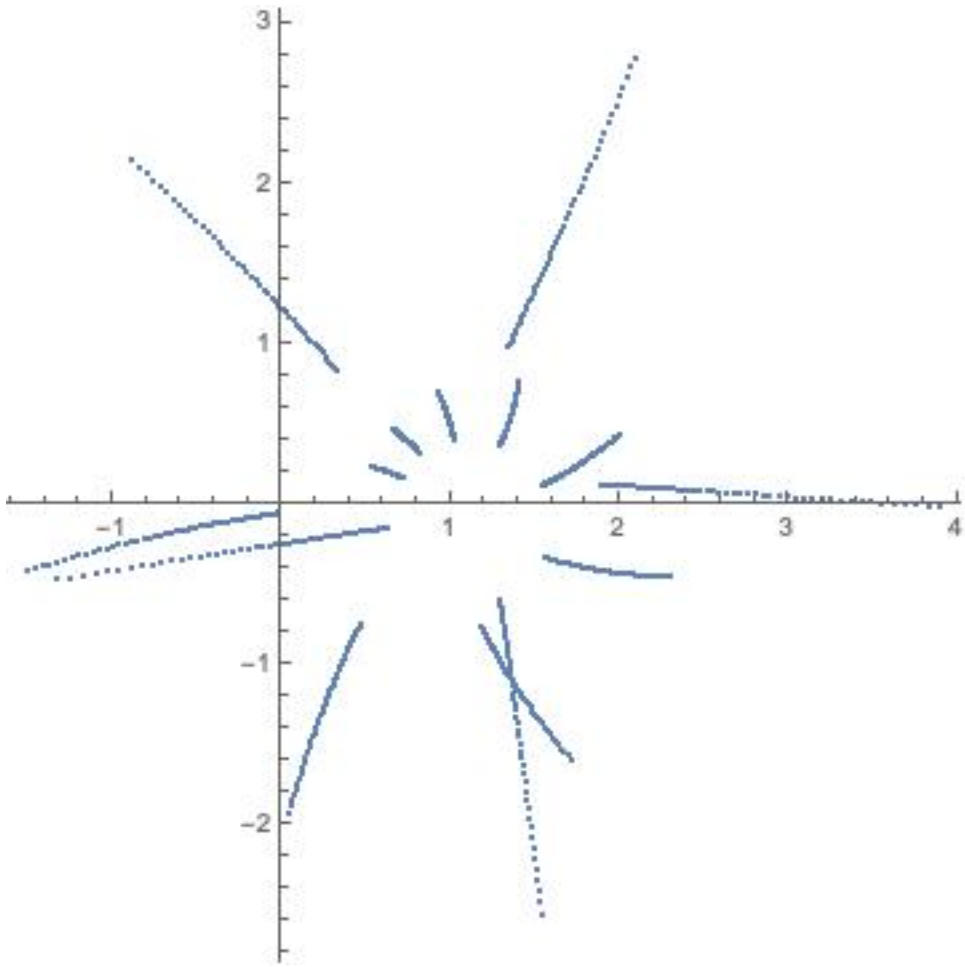
Lemma 2: if s_0 is a zero of eta then so is $1 - s_0*$.

If s_0 is a zero of eta then the eta reflection formula implies that $1 - s_0$ is also a zero of eta. By Lemma 1 this means that $1 - s_0*$ is also a zero of eta.

The modified reflection formula can be expressed: if $\sigma_0 + t_0 i$ is a zero of eta then so is $1 - \sigma_0 + t_0 i$. These two expressions have the *same* imaginary parts and that is key to what follows.

Visualization of the genesis of zeroes

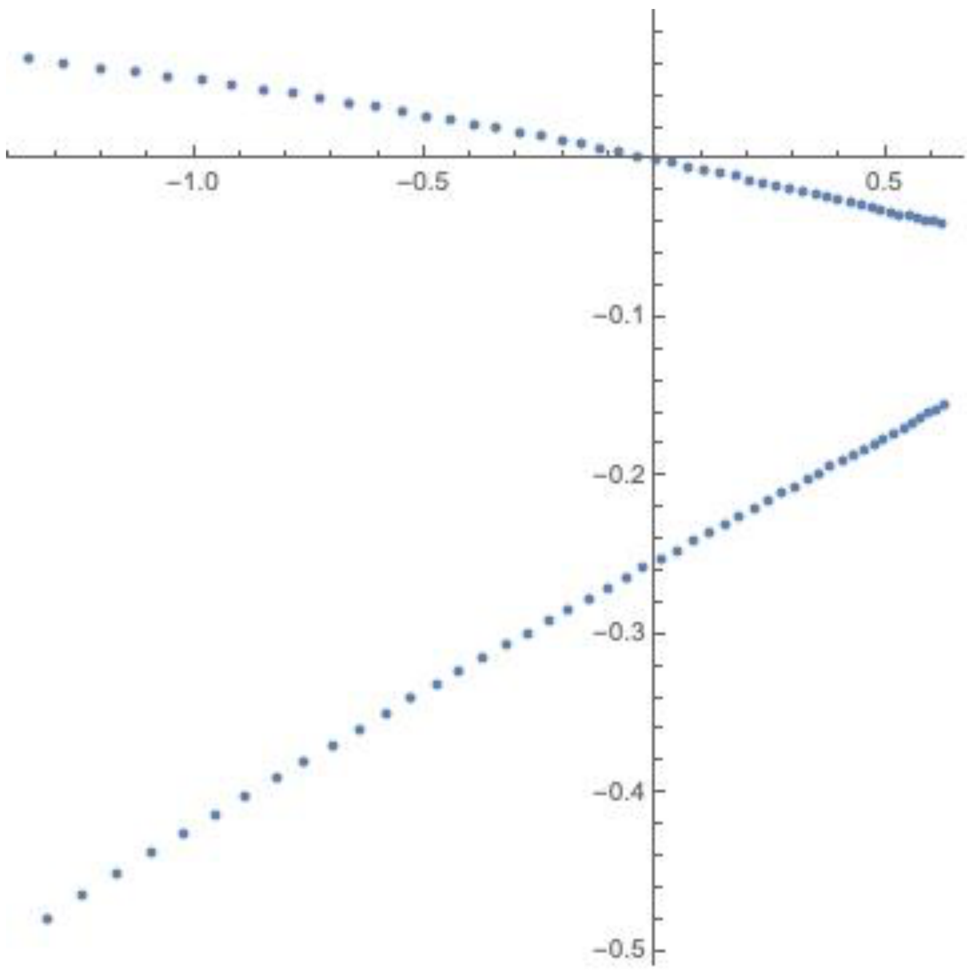
Displaying the behavior of eta visually greatly enhances our ability to understand how and why the non-trivial zeroes of eta (zeta) occur for $\sigma = 0.5$. We begin by restricting the argument of eta to the real axis of the critical strip. For $\sigma \in (0, 1)$ we find $\eta(\sigma) \in \big(0.5, ln(2)\big)$. The unit interval for σ is compressed by η into an interval of length 0.193…. Because η is continuous and infinitely differentiable in both σ and t, we expect that small variations in t will create small variations in η. We know that the first zero of eta has an imaginary part slightly bigger than 14. Therefore we will begin by looking at what happens to a set of points for $\sigma = .02, .04, .08, \dots, .98$ (steps of .02) and for a fixed value of t. The choice of discrete values of σ is imposed in order to keep the magnitude of the computation reasonable. For the first figure below, the choice of spacing works pretty well and the sets of points for a fixed t represent a *string* very well. Some discreteness does appear in the longer segments (larger t values) and shows a non-uniform density.



For users of Mathematica 12 this figure is the output of a manipulation of the program written as

Table[Sum[((-1)^(n - 1))*(1/((n)^sig))*Exp[-t*Log[n]*i],{n, 1, ∞}], {sig, .02, .98, .02}, {t, 1, 14, 1}]

There is a lot of information in this figure. Each string (I allow a string to have curvature, a slight amount in this figure, and much more in later figures) is labeled by t and is made up of the values of eta for t and for 49 evenly spaced values of σ with the 25^{th} value equal to 0.5. There are 14 strings in the figure corresponding to 14 values of t from 1, 2, 3, … , to 14. The $t = 1$ string is located near the 1 on the abscissa at about 10 o'clock. It is the shortest string in the plot, slightly larger than 0.193 (the arc length for the case $t = 0$). As we go clock-wise we see the strings for t = 2, 3, 4, 5, and 6. Clearly they are getting longer in arc length and are moving away from the origin. String 6 appears below the abscissa at about 4 o'clock. The points on the string are arranged with the small values of σ distal to the 1 on the abscissa and the larger values of σ proximal to the 1 on the abscissa. Since the eta values for uniformly spaced σ's are not uniformly spaced the $\sigma = 0.5$ value of eta is not halfway along the length of the string. By enlarging the figure the location of this special σ can be found. The 9th and the 14th strings are close to each other just before 9 o'clock. The 9th string is the shorter of the two. It's $\sigma = 1$ value (actually 0.98 in this plot) is closest to the origin. It looks like a slight adjustment in the t value would put the $\sigma = 1$ right on the origin. Indeed, $t = 9.0647\ldots$ does do so. This corresponds precisely with a trivial zero given above by $1 - i\frac{k2\pi}{\ln(2)}$ for the case of $k = 1$. Similarly the 14th string looks like an adjustment of its t value would possibly put $\sigma = 0.5$ on the origin. Let us try $t = 14.134725\ldots$. This is plotted below.



The lower string is the $t = 14$ string from the previous plot. The difference in apparent slope is the result of different aspect ratios in the two plots. The upper curve is for $t = 14.134725$, the imaginary part of first non-trivial zero for eta. Moreover, $\sigma = 0.5$ corresponds with the 25th dot in the 49 dot representation of the σ interval (it can be located by counting from the left). As you can see the typical value for a dot is a real part in the tenths and an imaginary part in the tenths as well. However the 25th σ dot has the eta value $1.62123\times10^{-6} - 2.6635\times10^{-7}\, i$. Since we have expressed the t for the first zero of eta to only one part in 10^{-6} we cannot expect to get "zero" to any better precision. The smaller values of σ produce the points to the left in the figure and the larger values of σ produce the points to the right in the figure.

The question that shouts out from this account is why does the string intersect the origin at $\sigma = 0.5$?! This is where the modified reflection formula comes into play.

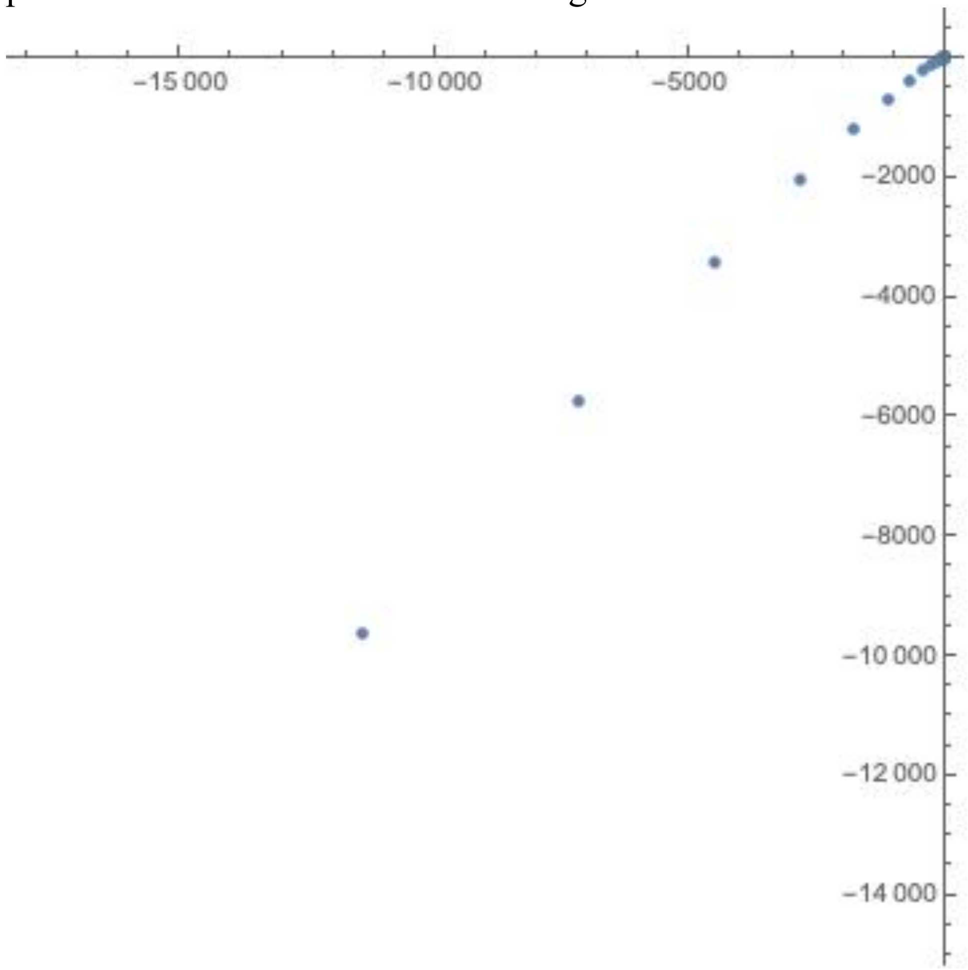
Theorem:
A string labeled by t can intersect the origin only for $\sigma = 0.5$.

Suppose that a string labeled by t' does intersect the origin for the value σ'. Therefore $\sigma' + t'i$ is a zero of eta. The modified reflection formula states that $1 - \sigma' + t'i$ is also a zero. Since these two zeros have the same t' they are on the same string. No string can intersect a point (the origin) more than once. Thus the two points must be the same point: $\sigma' + t'i = 1 - \sigma' + t'i$ which implies $\sigma' = 0.5$.

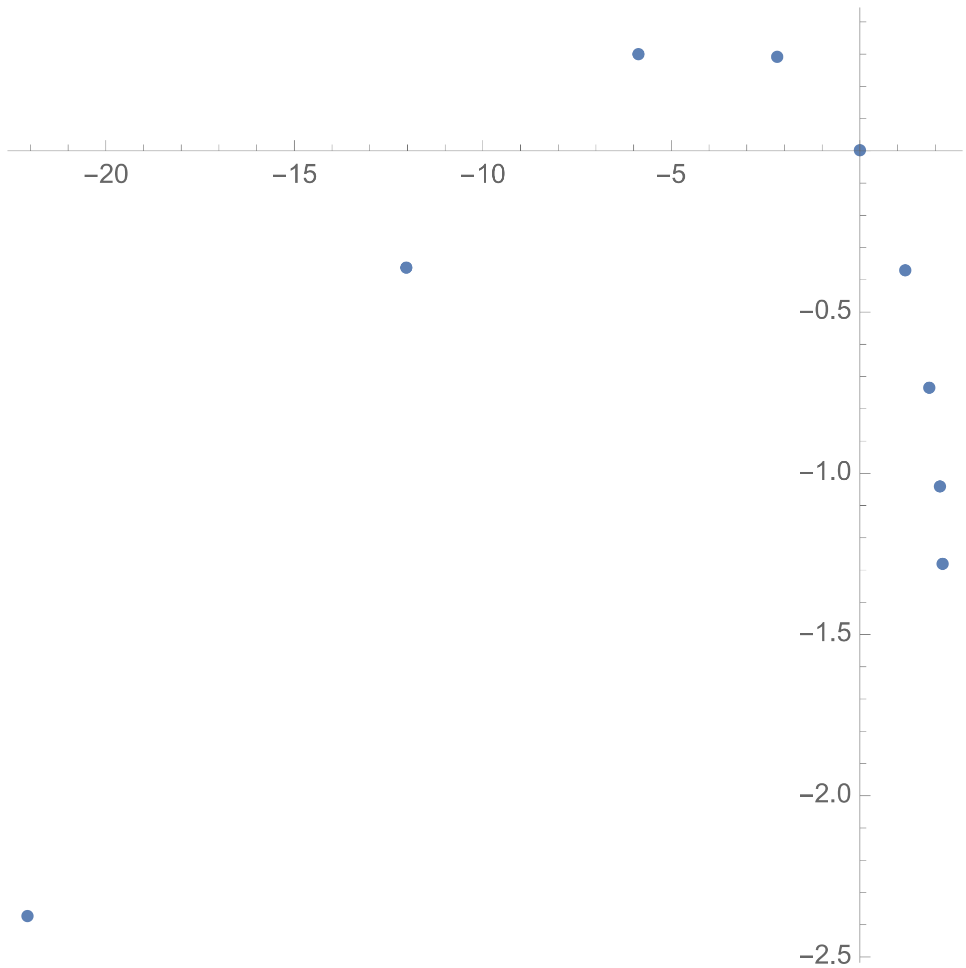
There is one possibility that would invalidate this conclusion. What happens if the string intersects itself? That is suppose that $\sigma' + t'i$ and $1 - \sigma' + t'i$ are zeroes of eta and $\sigma' \neq 0.5$. Then the string intersects itself at the origin producing a loop there. Thus the Riemann hypothesis would be false. To investigate this possibility further we will have to look at the behavior of strings for much larger values of t, because for the small values exhibited in the figures so far there is no reason to suspect self-intersections.

Complexity of strings for large t

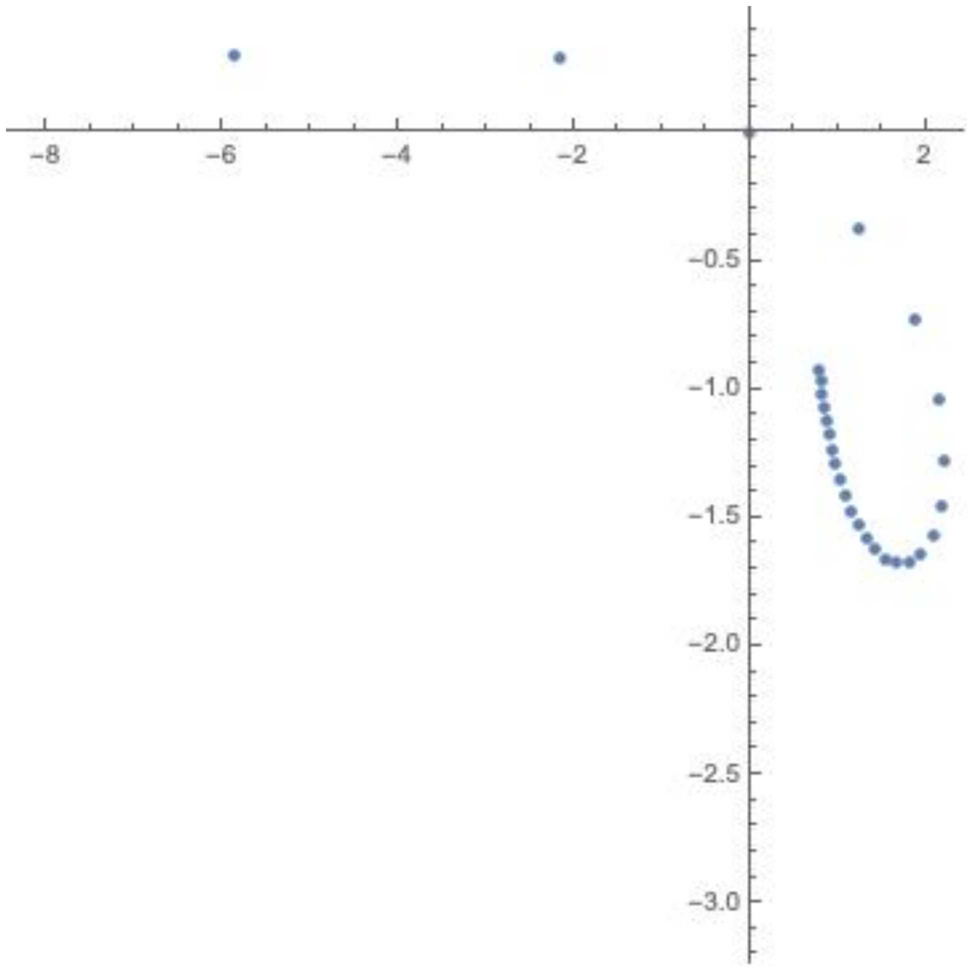
Let us begin our study of large t strings with the value $t = 267653395648.8475231278$. This has 22 digits, 12 to the left of the decimal point and 10 to the right. We have it on good authority that this is the imaginary part of a zero [4]. If we compute the eta values for all 49 σ points and for this value of t then we get



Note the scale. The arc length is more than 14,000. The point distal to the origin is the eta value for $\sigma = .02$. After the first 7 discrete points the remaining 42 points are a smear. By restricting σ to the values (.42, .44, .46, .48, .5, .52, .54, .56, .58) we obtain a blow-up of the region around the origin. The point at the origin has the value $0.000939283 + 0.00431777i$ which is incredibly small compared to the other values of points in the plot.



The small values of σ make up the points on the left branch whereas the large values of σ make up the points on the right branch. Unlike before it appears that the large σ values of eta are receding from the origin like the small σ values but in a different direction. This is incorrect as can be seen by including more points. If we include σ values from .38 to .98 in steps of .02 then we get

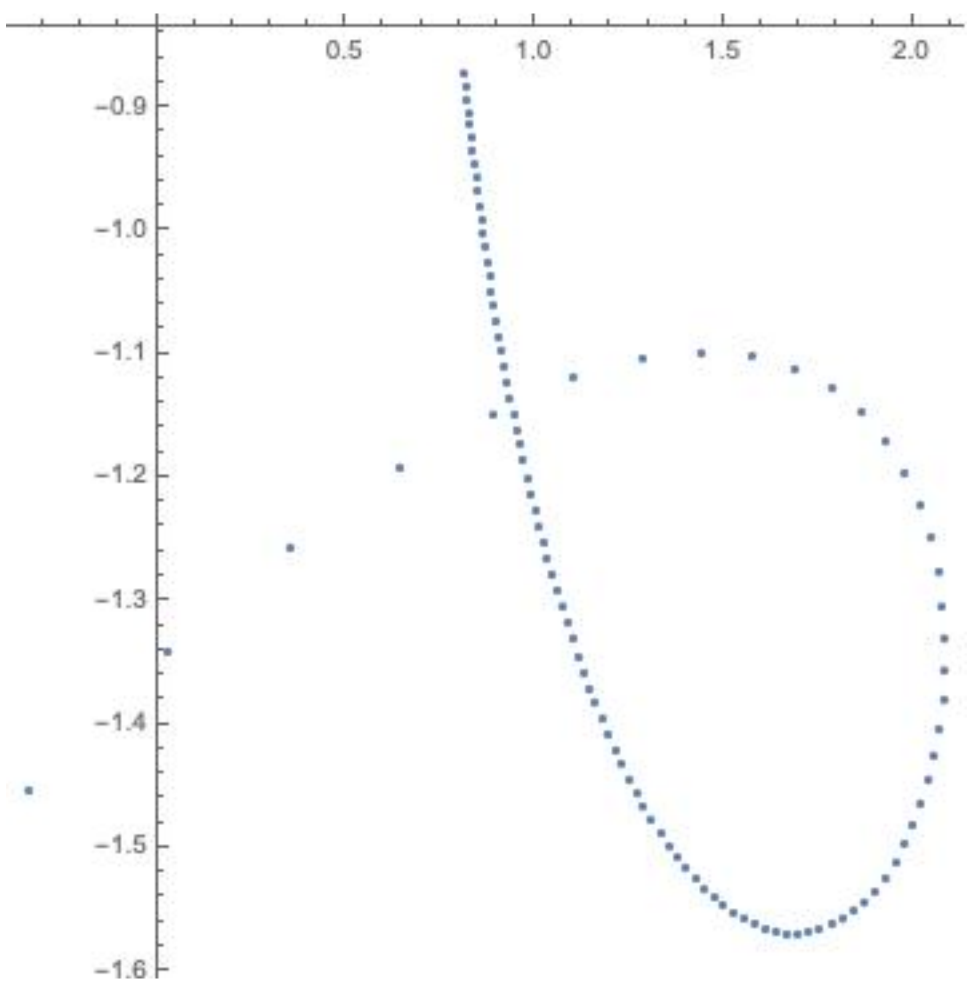


The right branch does not go very far out but instead turns inward making the whole plot a kind of logarithmic spiral. You can count the 24 dots to the right of the origin. On the left there are 22 more dots unseen because they are too far out for the scale of this plot (they can be seen in the first plot for this value of t).

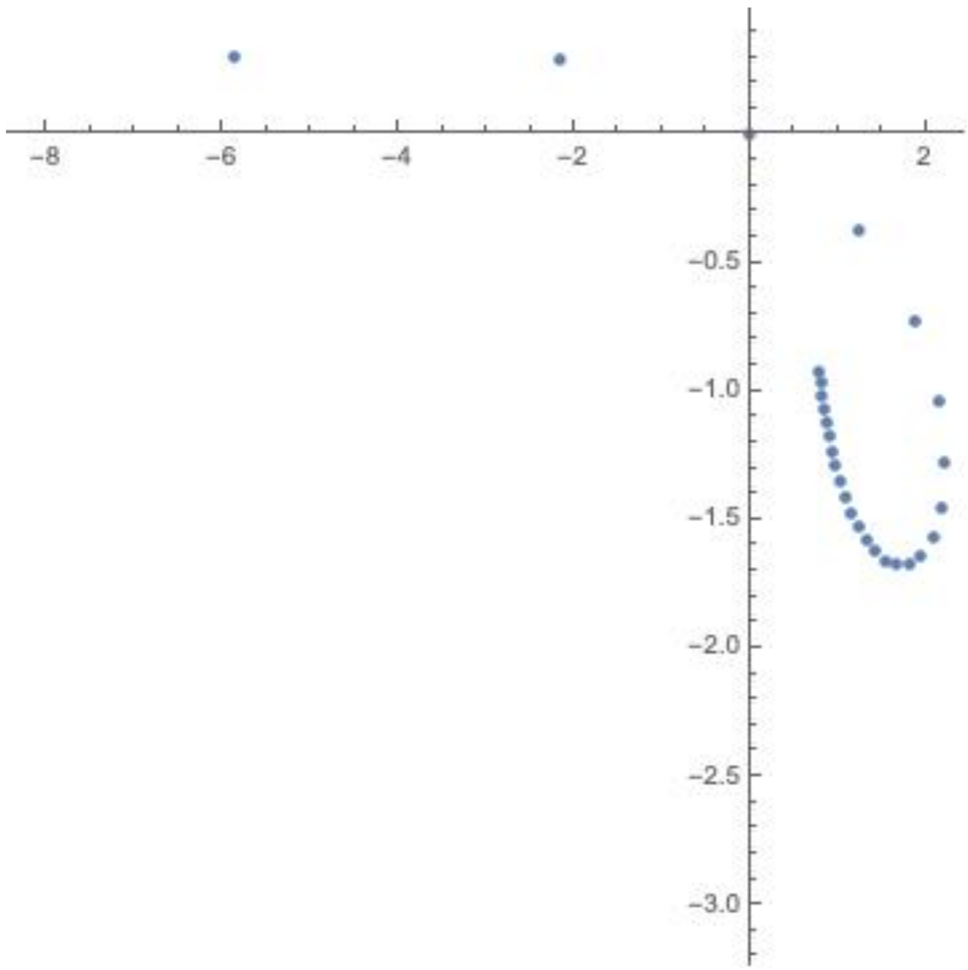
Note that we have referred to $t = 267653395648.8475231278$ as a large value of t. That is a matter of perspective. Clearly there are still larger values with zillions of digits. What do the strings look like for such larger values, that are still finite and therefore not really so big. Does the logarithmic spiral like structure seen above develop more turns but remain non-self-intersecting, implying that the Riemann hypothesis is true, or do self-intersections begin to occur, at the origin, for some large threshold value of t, implying that the Riemann hypothesis is false?

Observation of self-crossing t-strings

If it could be proved that t-strings are never self-crossing then the Riemann hypothesis (RH) is true, following from the theorem above. All my attempts at a proof failed. Finally the contrary position was adopted and it did not take long to find a counter-example. From Odlyzko's tables [4] of large t's we chose $t = 267653395648.8475231278$ to use above. In the first figure above where small values of t, 1, 2 … 14, are looked at the t-strings are slightly curved. For the large Odlyzko t values, such as the one we have selected, one end of the t- string is tightly wound (the end emanating from $0.5 < \sigma < 1$). By moving t slightly away from its value as the imaginary part of an eta zero we get a crossing:

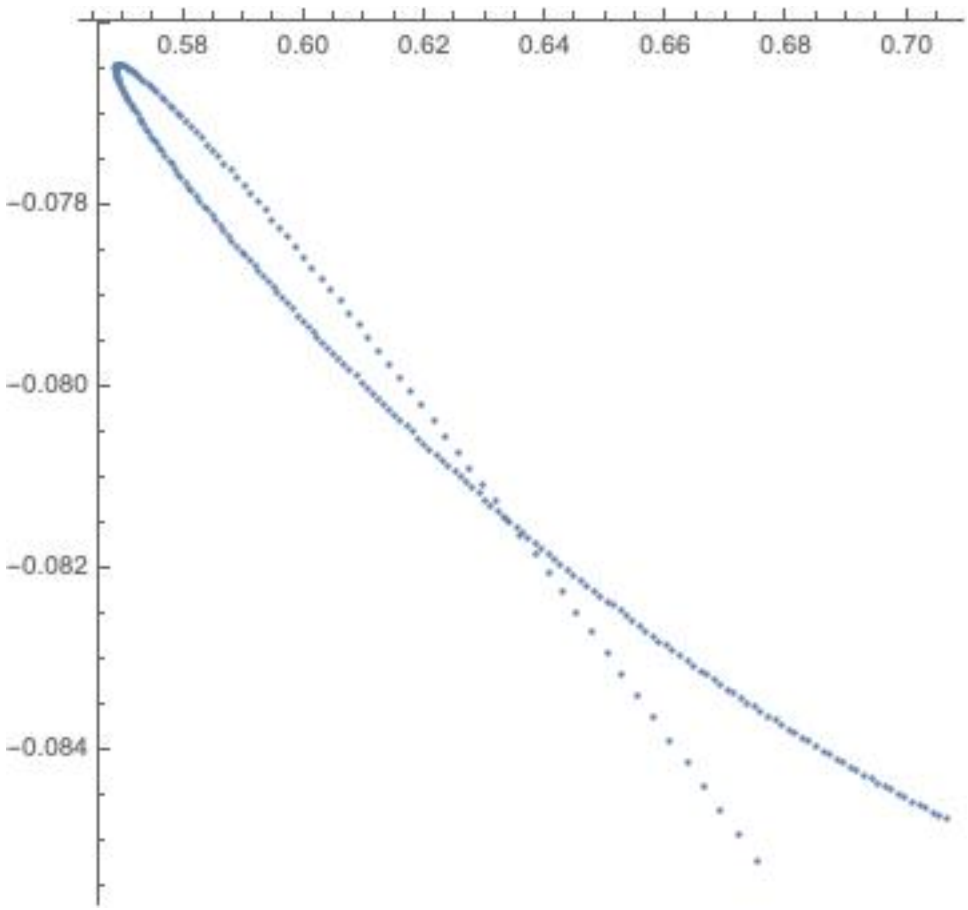


The value of t is 267653395648.83. The left branch (the end emanating from $0 < \sigma < 0.5$) shoots down to $-199897 - 605732\, i$ that is too big to be plotted with the loop crossing that occurs for large σ's. Note that for the overall scale of this t- string the crossing occurs rather close to the origin. The dots correspond with choosing σ to go from 0.005 to 1.0 in steps of 0.005. This plot only uses σ from 0.495 to 1.0 in steps of 0.005 so that the long tail does not dominate the plot. There are 102 dots. The dot next to the ordinate at -1.34 is the dot for $\sigma = 0.5$. Clearly the crossing is for two different values of σ. Indeed the crossing points are both for values of σ that are bigger than $\sigma = 0.5$. Moreover, the use of Odlyzko's zero, unchanged, clearly shows that the origin is crossed for $\sigma = 0.5$

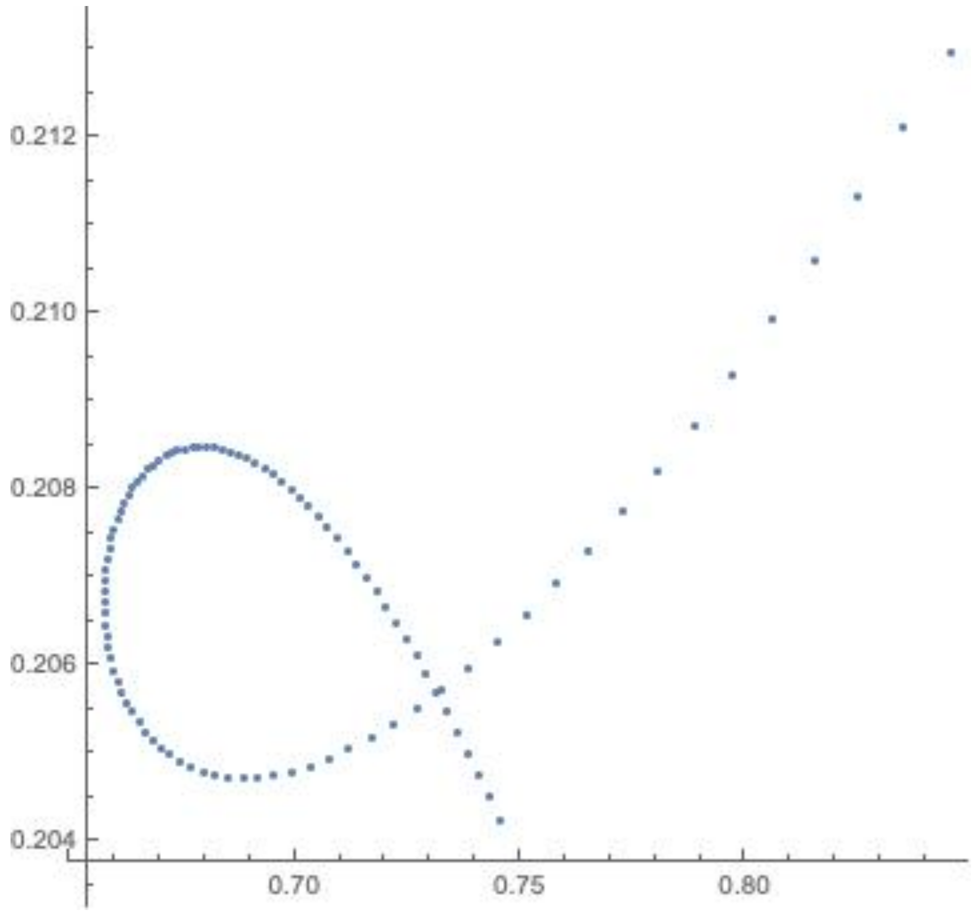


The spacing of the dots is 4 times coarser in this figure compared to the one above it. The increase in the value of t for this figure (a zero), needed to get the value for t in the preceding figure (a crossing), is just 0.0175231278 out of $2.6765339564883\times10^{11}$. However, for this value of t (a zero) there is *no longer* a self-crossing of the t-string.

After making a more extensive search for examples of crossings it was found that t needn't be so large. For $t = 231.61$ a tight hairpin crossing occurs. The figure below shows this feature for σ from 0.5 to 1 in steps of 0.005. This crossing value, $t = 231.61$, is about midway between the imaginary parts of two zeros: 231.250188700 and 231.987235253. However at the t values for the two zeros there are no crossings.



Another example of a robust crossing for small t is for $t = 357.60$, about half way between the imaginary parts of two zeros: 357.151302252 and 357.952685102. The features are the same as above, σ from 0.5 to 1 in steps of 0.005. However at the t values for the two zeros there are no crossings.



Discussion

Had we been able to prove that every t- string never has a self-crossing then we would have a proof of the RH. That these curves can self-intersect has been demonstrated by examples. This is not equivalent to a disproof of the RH. Only if we can demonstrate a self-intersection coincidently at the origin as well, would we have a disproof (a zero and two values of $\sigma \neq 0.5$). So far we have an example of a self-crossing loop formation tantalizingly close to the origin. The crossing in the first crossing plot above, for large t, is located with a real part of size 0.9 and an imaginary part of size -1.15, whereas the arc-length of the entire t-string (not shown) is of order 6.3×10^5. Typically the small values of σ are transformed by eta into a very long tail with only slight overall curvature, and certainly no crossings. The effect of eta on large values of σ is to produce a tightly wound segment of very limited extent and sometimes with crossings. So far a crossing for a σ less than 0.5 has not been observed. Only a value slightly less than 0.5 would be possible for a crossing because the structure of the long, small σ, tails is incompatible with crossings. Such a crossing for σ slightly less than 0.5 at the origin would require a partner σ slightly larger than 0.5 according to the *modified reflection formula*. The loop formed in this way would contain the image of $\sigma = 0.5$ on its interior and not at the zero. If such an example can be found RH is false. If instead it can be shown that crossings never happen for $\sigma < 0.5$ then RH is true.

Analytically finding conditions for self-intersections of t- strings and then finding additional conditions for crossing points to be coincident with the origin may prove as demanding as, say, proving or disproving RH. Having found examples of self-crossings by accident does not mean that finding crossings at the origin will be as easy (they may not exist). Any mathematical analysis will involve logarithms, trigonometric functions and of course the prime numbers. Each of these is overtly evident in the eta function in the form of trigonometric functions of logarithms of primes. Complex analysis, algebra and differential/integral calculus play their roles as well. In this paper, however, the emphasis is on the computer based computational heuristics of t-strings. At first it was thought that proving that t- strings never intersect themselves would be enough, until it was realized that it is not true. Such intersections take place for large t and for small t. However, observations so far suggest that intersections never occur for the t values for zeros and zeros never occur for t values of intersections. Instead of proving there cannot be crossings, which is false, it is enough to prove that crossings never occur for t values of zeros or for $\sigma < 0.5$. This suggests that RH is true. What can happen when t is truly large, so that $\ln(\ln[t])$ is very large, say, is unknown. Possibly more ornate multiple crossings are possible.

RH has tempted many and many different approaches have been tried. Some approaches have led to novel developments in both old and new fields of mathematics. Numerous equivalent problems have been identified. Hundreds of incorrect analyses have been proffered. Even some great mathematicians have been stymied. For myself I have enjoyed this sojourn into the unfathomable, the ineffable and the arcane. Perhaps this heuristic visualization construct, the t-line-segment, will enable a few others to more clearly *see* the meaning of RH.

In the first figure 14 line segments have been plotted together. If t is interpreted as time then one can make a movie that shows the sequence of t-

strings one after the other. In reality the strings do not evolve from each other sequentially. Each emanates from a fixed t and the unit interval of σ's. However, a movie would illustrate how the line segments smoothly adjust as zero crossings occur and as self-crossings come and go.

References

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