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Entropy evolution for the Baker map

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Gibbs entropy is invariant for the Baker map. A Jordan basis spectral decomposition of the Baker Frobenius-Perron operator suggests that any initial density evolves to the stationary density that has maximal entropy. This entropy conundrum is resolved by considering the difference between weak and strong convergence. A binary representation is used to make these points transparent. © *1998 American Institute of Physics.* [S1054-1500(98)02002-3]

The study of simple dynamical systems such as maps can clarify the understanding of phenomena in much more complicated and realistic systems. The Baker map has been studied for this reason. It is an invertible, chaotic map. Gibbs entropy is conserved during time evolution generated by the Baker map. However, construction of a Jordan basis for the Baker Frobenius-Perron operator and construction of the spectral decomposition for this operator strongly suggest that the Baker map takes any initial probability density to the stationary density that has maximal entropy. This is in conflict with the conservation of Gibbs entropy. In this paper, it is shown that this conflict stems from the difference between weak and strong convergence, and that the limit of the entropy and the entropy of the limit are not equal. These features are made transparent through use of a binary representation. The results clarify understanding of the approach to the Gibbs microcanonical ensemble in real classical mechanical systems.

I. INTRODUCTION

In a recent paper,[1] the construction of the Jordan basis for the Baker map was given by a straight-forward recursion formula. With this basis, a spectral decomposition of the Frobenius-Perron operator for the Baker map is easily constructed. This decomposition makes the decay of correlations for the invertible, chaotic Baker map patent. Nevertheless, it produces the conundrum that apparently any initial probability density decays to the invariant density, equal to the constant 1 on the unit square for the Baker map, while the Gibbs entropy is known to be an invariant. Since the primary purpose of the study of simple maps is to lay bare the dynamical details, this paper is written to explicitly demonstrate that no inconsistency exists. The resolution of the conundrum involves consideration of the differences between weak and strong convergence. Using the spectral decomposition of the Frobenius-Perron operator and an associated binary representation, we show that the limit of multiple iterations of the Frobenius-Perron operator on the probability density is the key. The limit of the entropy and the entropy of the limit are not equal.

These ideas are not new. They have their origin in earlier work on Pollicot-Ruelle resonances.[2] The extension of these ideas to discrete maps, including the Baker map, has been treated extensively by several authors.[3,4] These papers contain the spectral decomposition of the Frobenius-Perron operator and a discussion of the decay of correlations, as well as the issue of convergence. Elsewhere,[5] the time evolution of the Gibbs entropy for the Baker map has been explored and its invariance has been demonstrated. The direct connection with the problem of convergence has also been elucidated.[5] Resolution of the apparent conflict[1] of the spectral decomposition perspective with the entropy evolution perspective is the purpose of this paper.

Mackey has shown[5(a)] that the Baker map is mixing, that a necessary and sufficient condition for mixing is weak convergence to a unique stationary density for all initial densities, that correlations decay, but that *mixing is not sufficient to ensure the convergence of the entropy to a maximum.*

The Baker map acts on the unit (x,y)-square. Denote a probability density on the unit square by $f(x,y)$. The Gibbs entropy functional, $S[f]$, is defined by[5(a)]

$$S[f]=-\int_0^1 dx\int_0^1 dy\, f(x,y)\ln[f(x,y)]. \tag{1}$$

Denote the Frobenius-Perron operator for the baker map by P_{Baker}. The invariance of the Gibbs entropy[1,5(a)] may be expressed by

$$S[P_{\text{Baker}}f(x,y)]=S[f(x,y)]. \tag{2}$$

Denote the Jordan basis for the Baker map by[1] $|p,\nu\rangle_J$. Express an initial probability density by the expansion

$$f(x,y)=\sum_{p=0}^{\infty}\sum_{\nu=0}^{p} C_{p\nu}|p,\nu\rangle_J. \tag{3}$$

It has been shown[1,4(b)] that the nth iterate of the Baker Frobenius-Perron operator on a Jordan state produces the formula

$$P^n_{\text{Baker}}|p,\nu\rangle_J=\sum_{k=0}^{[n,p-\nu]_{\min}}\frac{n!}{k!(n-k)!}\frac{1}{2^{(n-k)p}}|p,\nu+k\rangle_J. \tag{4}$$

Since $|0,0\rangle_J=1$ for all x and y, the integral of $f(x,y)$ over the unit square equals 1 since $f(x,y)$ is a probability density, and[1]

$$\int_0^1 dx\int_0^1 dy|p,\nu\rangle_J=\delta_{p0}\delta_{\nu 0}; \tag{5}$$

it follows that

$$C_{00}=1. \tag{6}$$

Therefore, Eqs. (3), (4) and (6) combine to imply

$$\underset{n\to\infty}{\text{Limit}}P^n_{\text{Baker}}\,f(x,y)=|0,0\rangle_J. \tag{7}$$

Therefore,

$$S[\underset{n\to\infty}{\text{Limit}}P^n_{\text{Baker}}\,f(x,y)]=0\geqslant S[f(x,y)], \tag{8}$$

with a strict inequality if

$$f(x,y)\neq|0,0\rangle_J. \tag{9}$$

Thus, for any initial density other than the invariant density, $|0,0\rangle_J$, *it would appear that the entropy does converge to a maximum*. This is the conundrum referred to above.

The resolution of this conundrum stems from the imprecise use of the concept of limit in Eq. (7). We show below that convergence is in the weak sense and not in the strong sense. The entropy of the weak limit does not equal the limit of the entropy. These facts are most transparently seen using a binary representation for a special case.

II. BINARY REPRESENTATIONS

The Frobenius-Perron operator for the Baker map has the explicit form[1,5(a)]

$$P_{\text{Baker}}f(x,y)=f\left(\frac{x}{2},2y\right)\Theta\left(\frac{1}{2}-y\right)+f\left(\frac{x}{2}+\frac{1}{2},2y-1\right)\Theta\left(y-\frac{1}{2}\right), \tag{10}$$

where Θ is the Heaviside function. Let the set A be defined by

$$A=\{(x,y)|y<\tfrac{1}{2}\} \tag{11}$$

and denote the characteristic function on A by 1_A. Choose as the initial probability density

$$f(x,y)=2\times 1_A. \tag{12}$$

Clearly, this initial density has the value 2 in the lower half of the unit (x,y)-square and the value 0 in the upper half and is normalized to unity. An application of the Baker Frobenius-Perron operator, according to Eq. (10), produces a density with four horizontal strips in the unit (x,y)-square, two with the value 2 for all x and with $0\leqslant y<1/4$ or with $1/2\leqslant y<3/4$, and two with the value 0 for all x and with $1/4\leqslant y<1/2$ or with $3/4\leqslant y<1$. Clearly, the next iteration of the operator will produce eight alternating horizontal strips of 2's and 0's. What is the limit of this iteration process? One might guess that the answer is infinitely many infinitesimally thin alternating strips of 2's and 0's. The correct answer, however, is the stationary density that equals 1 everywhere on the unit square. This will be demonstrated below.

In Ref. 1, we showed that the Jordan basis for the Baker map can be constructed from expansions in terms of product states denoted by $|m,n\rangle$ that are defined by

$$|m,n\rangle=R_m(x)L_n(y), \tag{13}$$

in which $R_m(x)$ and $L_n(y)$ are, respectively, proportional to the eigenfunctions of the Frobenius-Perron operator for the Bernoulli map and the Koopman operator for the Bernoulli map. The corresponding adjoint states are denoted by $\langle j,k|$, are given by[1]

$$\langle j,k|=L_j(x)R_k(y), \tag{14}$$

and satisfy biorthogonality,

$$\langle j,k|m,n\rangle=\delta_{jm}\delta_{kn}. \tag{15}$$

Thus, instead of an expansion in terms of the Jordan basis, we can treat this special case with an expansion in terms of these product states,

$$2\times 1_A=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_{mn}|m,n\rangle. \tag{16}$$

Equations (15) and (16) imply

$$C_{mn}=\langle m,n|2\times 1_A\rangle=2\int_0^1 dxL_m(x)\int_0^{1/2}dyR_n(y)$$
$$=\delta_{m0}\int_0^1 dyR_n(y/2), \tag{17}$$

in which an integral property[1] of the $L_m(x)$'s and a change of variables from y to $y/2$ have been used. This result implies that the expansion given in Eq. (16) can be simplified to

$$2\times 1_A=|0,0\rangle+\sum_{n=1}^{\infty}C_{0n}|0,n\rangle. \tag{18}$$

The state $|0,0\rangle$ is identical[1] with the Jordan state $|0,0\rangle_J$. The normalization of the initial state given by Eqs. (12) and (18) is accounted for by the $|0,0\rangle$ term alone since the $|0,n\rangle$ states each integrate over the unit square to zero.[1] While we can give an explicit expression for generation of the coefficients, C_{0n}, their explicit values will not be required for the rest of this discussion. The $|0,n\rangle$ states have a very nice property;[1] they are eigenstates of the Baker Frobenius-Perron operator, i.e.,

$$P_{\text{Baker}}|0,n\rangle=\frac{1}{2^n}|0,n\rangle. \tag{19}$$

This is a very special case of a much more complicated formula[1] for the application of P_{Baker} to general product states $|m,n\rangle$ with $m\neq 0$ that are not generally eigenstates. Clearly, we now get

$$P^n_{\text{Baker}} 2\times 1_A = |0,0\rangle + \sum_{k=1}^{\infty} C_{0k}\frac{1}{2^{nk}}|0,k\rangle, \tag{20}$$

wherein we have changed the summation index n of Eq. (18) to k in this equation. It is now apparent that

$$\underset{n\to\infty}{\text{Limit}}\; P^n_{\text{Baker}} 2\times 1_A = |0,0\rangle, \tag{21}$$

and we have again obtained the entropy conundrum: the entropy of this limit is maximal with the value 0, whereas the entropy of the initial state is $-\ln(2)$, but the Baker Frobenius-Perron operator preserves the entropy!

What is happening here? For any finite value of n, the state on the right-hand side of Eq. (20) has a graph on the unit square made up of 2^{n+1} alternating horizontal strips with values of 2 or 0 as was discussed above. It is easy to verify that the entropy for any of these states is $-\ln(2)$, the invariant value corresponding with the initial value. Only in the limit does the value of the entropy change discontinuously to the value 0. To see how this comes about, we use a binary representation. Let $.b_1b_2b_3\ldots$ denote a general binary representation of a number y in the unit interval. The number y is given explicitly by

$$y = \sum_{j=1}^{\infty}\frac{b_j}{2^j}, \tag{22}$$

where the b_j's are either 0 or 1. Thus, the function $2\times 1_A$ has the value 2 for all binary representations expressible as $.0b_2b_3b_4\ldots$ and the value 0 for all those expressible as $.1b_2b_3b_4\ldots$ where b_2, b_3, b_4, etc. may be either 1 or 0. We write this in the form

$$2\times 1_A = \left\{\begin{matrix} 2, & \text{for } .0b_2b_3b_4\ldots \\ 0, & \text{for } .1b_2b_3b_4\ldots \end{matrix}\right\}. \tag{23}$$

Clearly, an application of the Baker Frobenius-Perron operator produces

$$P_{\text{Baker}} 2\times 1_A = \left\{\begin{matrix} 2, & \text{for } .00b_3b_4b_5\ldots \\ 2, & \text{for } .10b_3b_4b_5\ldots \\ 0, & \text{for } .01b_3b_4b_5\ldots \\ 0, & \text{for } .11b_3b_4b_5\ldots \end{matrix}\right\}. \tag{24}$$

Notice that the distinction between the values 2 and 0 resides in the second binary position as opposed to the first binary position for Eq. (23). A second application of the operator yields

$$P^2_{\text{Baker}} 2\times 1_A = \left\{\begin{matrix} 2, & \text{for } .000b_4b_5b_6\ldots \\ 2, & \text{for } .010b_4b_5b_6\ldots \\ 2, & \text{for } .100b_4b_5b_6\ldots \\ 2, & \text{for } .110b_4b_5b_6\ldots \\ 0, & \text{for } .001b_4b_5b_6\ldots \\ 0, & \text{for } .011b_4b_5b_6\ldots \\ 0, & \text{for } .101b_4b_5b_6\ldots \\ 0, & \text{for } .111b_4b_5b_6\ldots \end{matrix}\right\}. \tag{25}$$

Now the distinction between the values 2 and 0 resides in the third binary position. The first two binary positions in this case are identical for values 2 and 0 and range over all possible (2^2) two digit binary expansions. It should now be clear what happens as we continue to iterate the operator. After n iterations, the initial n binary digit segments will be identical for both values 2 and 0 and will exhaust the 2^n possibilities, while the $n+1$ binary digit will distinguish the two cases, and all succeeding binary digits may have all possible values. For finite but arbitrary n, a distinction remains (in position $n+1$) that reflects the alternating strip structure of the graph of the function. However, in the infinite limit, all possible binary digit sequences occur for the value 2 and all possible binary digit sequences *also* occur for the value 0. The function has the averaged value, 1, at every point y in the unit interval; thus the function is equal to 1 on the entire unit square. The limit expressed by Eq. (21) is indeed obtained.

A sequence $\{f_n\}$ of L^1 functions is said to be *weakly convergent* to an L^1 function f if

$$\underset{n\to\infty}{\text{Limit}}\int_0^1 dy\, f_n(y)g(y) = \int_0^1 dy\, f(y)g(y), \tag{26}$$

for all $L^\infty g(y)$'s.[5(a)] By studying the behavior for arbitrary characteristic functions 1_B, where B is any measurable set in the unit square, in place of the g's, it is possible to prove that Eq. (21) is meaningful as weak convergence. In essence, very fine alternating strips of 2's and 0's integrated against a function g yield the same result as integrating g against the constant 1.

A sequence $\{f_n\}$ of L^p functions is said to be *strongly convergent* to an L^p function f if

$$\underset{n\to\infty}{\text{Limit}}\; \|f_n - f\| = 0, \tag{27}$$

wherein $\|\cdots\|$ denotes the L^1 norm.[5(a)] For the sequence of functions given in Eq. (20) as the f_n's and for the limit function given in Eq. (21) as the f, we find instead

$$\underset{n\to\infty}{\text{Limit}}\|f_n - f\| = 1. \tag{28}$$

Thus, we do not have strong convergence.

Equation (21) is meaningful in the sense of weak convergence which means only when both sides are used inside an integral. Weak convergence permits us to use the spectral decomposition of the Frobenius-Perron operator in correlation functions where an integral is performed as part of the definition of the correlation function.[3(c),4(b)] This makes the decay of correlations manifest. The density does converge to the invariant density, but only in the weak sense, so that the limit of the entropy equals the initial entropy, but is unequal to the entropy of the weak limit.

The behavior illuminated by the Baker map clarifies the Gibbs picture of the time evolution of a phase space distribution for a classical mechanical system. Gibbs' picture is that any initial distribution evolves to an intricate, highly filamentous structure that ultimately becomes dense on the energy surface. But it does not literally become the Gibbs microcanonical ensemble that uniformly covers the energy surface! If it did, then there would be a discontinuous jump in the Gibbs entropy in the limit. Thus the approach to the microcanonical ensemble is *weak convergence*, and requires that we use the microcanonical ensemble in integrals to compute expectation values and correlations.

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