The generalized Langevin equation with Gaussian fluctuations

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It is shown that all statistical properties of the generalized Langevin equation with Gaussian fluctuations are determined by a single, two-point correlation function. The resulting description corresponds with a *stationary, Gaussian, non-Markovian* process. Fokker–Planck-like equations are discussed, and it is explained how they can lead one to the erroneous conclusion that the process is *nonstationary, Gaussian, and Markovian*.

I. INTRODUCTION

The generalized Langevin equation provides a stochastic description of Brownian motion. In one dimension, it has the form

$$\frac{d}{dt}u(t) = -\int_0^t \beta(t-s)u(s)\,ds + \frac{1}{m}\tilde{f}(t) \tag{1}$$

in which $u(t)$ is the velocity of the Brownian particle at time t, m is its mass, $\beta(t-s)$ is the dissipative "memory kernel," and $\tilde{f}(t)$ is a Gaussian fluctuating driving force. It is assumed that $\tilde{f}(t)$ possesses first and second moments given by

$$\langle \tilde{f}(t)\rangle = 0 \quad \text{and} \quad \langle \tilde{f}(t)\tilde{f}(s)\rangle = k_B T m \beta(|t-s|), \tag{2}$$

in which k_B is Boltzmann's constant and T is the temperature of the fluid in which the Brownian particle is immersed. A Markovian limit of this description is obtained when $\beta(t-s) = 2\beta\delta(t-s)$ in which β is a constant and $\delta(t-s)$ is the Dirac delta function.

In recent papers, Adelman[1] and Fox[2] have derived Fokker—Planck-like equations corresponding to the process described by (1) and (2). It was even asserted,[2] on the basis of the Fokker—Planck-like equation, that the process being described must be a *nonstationary, Gaussian, Markovian* process. Here, it will be shown that the process is in fact a *stationary, Gaussian, non-Markovian* process, and that the Fokker-Planck-like equations of Adelman[1] and Fox[2] are not properly Fokker—Planck equations after all. It will be clearly indicated how the confusion arises, and the distinction between bona fide Fokker—Planck equations for bona fide *nonstationary, Gaussian, Markov* processes, and Fokker—Planck-like equations which arise in the study of *stationary, Gaussian, non-Markovian* processes will be elucidated.

II. THE SOLUTION TO THE GENERALIZED LANGEVIN EQUATION

Even though the process described by (1) and (2) will be seen to be non-Markovian, the Gaussian property of the fluctuating driving force leads to a complete stochastic description in terms of a single, two-point correlation function. Therefore, the Gaussianness provides a description which has a property usually associated with Markovianness, i.e., a single two-point function determines everything. In the case of Markovianness the two point function is the two-point conditional distribution P_2, which will be discussed below.

Using Laplace transforms and the definition

$$\hat{\beta}(z) = \int_0^\infty e^{-zt}\beta(t)\,dt \tag{3}$$

enables one to obtain the solution to (1) in the form

$$u(t) = \chi(t)u(0) + \frac{1}{m}\int_0^\infty \chi(t-s)\tilde{f}(s)\,ds, \tag{4}$$

in which $\chi(t)$ is defined through its Laplace transform

$$\hat{\chi}(z) \equiv [z + \hat{\beta}(z)]^{-1}. \tag{5}$$

While the Laplace transform method of treatment of the generalized Langevin equation is standard in the literature, some of the results to be given below appear not to have been previously published and greatly clarify the discussion.

In their pioneering work on Brownian motion, Uhlenbeck and Ornstein[3] observed that two types of averaging are necessary in a discussion of Brownian motion using the Langevin equation. The first type of averaging is with respect to the stochastic driving force, $\tilde{f}(t)$, and is denoted, as in (2), by $\langle\cdots\rangle$. The second type of averaging is with respect to the initial velocity $u(0)$, which appears in the solution (4) and will be denoted by $\{\cdots\}$. The distribution for $u(0)$ will be the Maxwellian

$$W(u(0)) = (2\pi k_B T/m)^{-1/2}\exp[-mu^2(0)/2k_B T]. \tag{6}$$

Using the solution (4), we can compute the velocity autocorrelation function for $t_2 \geq t_1$

$$\begin{aligned}\{\langle u(t_1)u(t_2)\rangle\} &= \chi(t_1)\chi(t_2)\{u^2(0)\} \\ &\quad + (k_B T/m)(\chi(t_2-t_1) - \chi(t_1)\chi(t_2)) \\ &= (k_B T/m)\chi(t_2-t_1).\end{aligned} \tag{7}$$

To get (7), we have used an identity which is proved in Appendix A, which states, for $t_2 \geq t_1$,

$$\Big\langle\Big(\int_0^{t_2}\chi(t_2-s_2)\tilde{f}(s_2)\,ds_2\Big)\Big(\int_0^{t_1}\chi(t_1-s_1)\tilde{f}(s_1)\,ds_1\Big)\Big\rangle = k_B T m(\chi(t_2-t_1) - \chi(t_1)\chi(t_2)) \tag{8}$$

and which is not found in the usual treatments of the problem by the Laplace transform methods. Equation (7) makes it quite plain that the process is *stationary*. For $t_2 = t_1$, (7) reduces to precisely the same result obtained from (6) for $\{u^2(0)\}$. Stationarity means that the Maxwellian persists.

Using the autocorrelation given by (7), we can construct the unconditioned two-point distribution function from the correlation matrix, by following a procedure discussed by Wang and Uhlenbeck.[4] The correlation matrix is

$$\frac{k_B T}{m}\begin{pmatrix} 1 & \chi(t_2-t_1) \\ \chi(t_2-t_1) & 1 \end{pmatrix} \tag{9}$$

and its inverse is easily seen to be

$$\left(\frac{k_B T}{m}\right)^{-1} \frac{1}{1-\chi^2(t_2-t_1)} \begin{pmatrix} 1 & -\chi(t_2-t_1) \\ -\chi(t_2-t_1) & 1 \end{pmatrix}. \tag{10}$$

This implies the two-point distribution function

$$W_2(u_1t_1;u_2t_2) = \left[\left(2\pi\frac{k_BT}{m}\right)^2 (1-\chi^2(t_2-t_1))\right]^{-1/2} \times \exp\left(-\frac{m(u_1^2+u_2^2-2u_1u_2\chi(t_2-t_1))}{2k_BT(1-\chi^2(t_2-t_1))}\right). \tag{11}$$

The validity of this result follows from the fact that $u(t)$, as given by (4), inherits the Gaussianness of $\mathcal{F}(t)$ as a consequence of linearity, and $\{\langle u(t)\rangle\} = 0$.

If we want the conditioned two-point distribution, then we can use the definition[4]

$$P_2(u_1t_1;u_2t_2) \equiv W_2(u_1t_1;u_2t_2)/W_1(u_1t_1). \tag{12}$$

However, from above we have $\{\langle u^2(t_1)\rangle\} = k_BT/m$, and with $\{\langle u(t_1)\rangle\} = 0$, it follows that

$$W_1(u_1t_1) = W_1(u_1) = (2\pi k_BT/m)^{-1/2} \times \exp(-mu_1^2/2k_BT). \tag{13}$$

As already mentioned, this persistence of the Maxwellian distribution exhibits the *stationarity* of the process. From (12) and (13) it follows that

$$P_2(u_1t_1;u_2t_2) = [2\pi(k_BT/m)(1-\chi^2(t_2-t_1))]^{-1/2} \times \exp[-m(u_2-\chi(t_2-t_1)u_1)^2/2k_BT(1-\chi^2(t_2-t_1))]. \tag{14}$$

Higher order distributions can also be constructed and they all depend upon $\chi(t-t')$, the two-point correlation function. In particular, the three-point, unconditioned distribution, $W_3(u_1t_1;u_2t_2;u_3t_3)$ for $t_3 \geq t_2 \geq t_1$ is determined from the corelation matrix

$$\frac{k_BT}{m}\begin{pmatrix} 1 & \chi(t_2-t_1) & \chi(t_3-t_1) \\ \chi(t_2-t_1) & 1 & \chi(t_3-t_2) \\ \chi(t_3-t_1) & \chi(t_3-t_2) & 1 \end{pmatrix} \tag{15}$$

by computing its inverse, a somewhat laborious but straightforward procedure.[4]

III. NON-MARKOVIANNESS OF THE SOLUTION

If the process were Markovian, then the Smoluchowski,[5] or Chapman–Kolmogorov,[6] equation would have to hold:

$$P_2(u_1t_1;u_3t_3) = \int_{-\infty}^{\infty} dv\, P_2(vs;u_3t_3)P_2(u_1t_1;vs) \tag{16}$$

for $t_3 \geq s \geq t_1$. For the result in (14), however, this requirement leads to the requirement

$$\chi(t_3-t_2)\chi(t_2-t_1) = \chi(t_3-t_1). \tag{17}$$

Equation (17) is only satisfied by

$$\chi(t-t') = \exp[-(t-t')D] \tag{18}$$

according to Doob's theorem.[7,8] But this implies, when (5) is used, that $\hat{\beta}(z) = D$, so that

$$\beta(t) = 2D\delta(t). \tag{19}$$

This is simply a Markovian limit of the generalized Langevin equation. Therefore, (17) is not satisfied and neither is (16). The process is *non-Markovian*. This is surely hardly a surprise given the presence of the "memory kernel" in (1).

IV. FOKKER-PLANCK-LIKE EQUATIONS FOR THE SOLUTION

Associated with (14) is the partial differential equation

$$\frac{\partial}{\partial t_2}P_2(u_1t_1;u_2t_2) = -\frac{\dot{\chi}(t_2-t_1)}{\chi(t_2-t_1)}\frac{\partial}{\partial u_2}(u_2P_2(u_1t_1;u_2t_2)) - \frac{k_BT}{m}\frac{\dot{\chi}(t_2-t_1)}{\chi(t_2-t_1)}\frac{\partial^2}{\partial u_2^2}P_2(u_1t_1;u_2t_2) \tag{20}$$

subject to the initial condition $P_2(u_1t_1;u_2t_1) = \delta(u_2-u_1)$. $\dot{\chi}(t_2-t_1)$ denotes the derivative of $\chi(\tau)$ with respect to τ. If we consider the special case $t_1=0, t_2=t, u_1=u(0)$, and $u_2=u$, then (20) looks like

$$\frac{\partial}{\partial t}P_2(u(0);ut) = -\frac{\dot{\chi}(t)}{\chi(t)}\frac{\partial}{\partial u}(uP_2(u(0);ut)) - \frac{k_BT}{m}\frac{\dot{\chi}(t)}{\chi(t)}\frac{\partial^2}{\partial u^2}P_2(u(0);ut), \tag{21}$$

with the initial condition $P(u(0);u0) \equiv \delta(u-u(0))$.

Equation (21) looks very much like a bona fide Fokker–Planck equation for a *nonstatinary*, *Gaussian*, *Markov* process and is precisely the equation both Adelman and Fox obtained earlier by a different procedure. Below, it will be shown that bona fide *nonstationary*, *Gaussian*, *Markov* processes do lead to Fokker–Planck equations of precisely the form of (21) *but* with *less* stringent initial conditions. It will also be shown that (21) will *not* lead to results consonant with (20) and (14) if it is treated as a bona fide Fokker–Planck equation. The reasons for these distinctions are manifest in (20) wherein the coefficients $\dot{\chi}(t_2-t_1)/\chi(t_2-t_1)$ exhibit explicit dependence on *both* t_2 and t_1.

A bona fide *nonstationary*, *Gaussian*, *Markov* process is described by the equation

$$\frac{d}{dt}u(t)=-\beta(t)u(t)+\frac{1}{m}\tilde{f}(t) \tag{22}$$

with a Gaussian fluctuating force $\tilde{f}(t)$ possessing first and second moments

$$\langle\tilde{f}(t)\rangle=0 \quad\text{and}\quad \langle\tilde{f}(t)\tilde{f}(s)\rangle=2k_BTm\beta(t)\delta(t-s). \tag{23}$$

The solution to (22) is

$$u(t)=\exp[-\int_0^t\beta(s)ds]u(0)$$
$$+\int_0^t\exp[-\int_s^t\beta(s')ds'][\tilde{f}(s)/m]ds. \tag{24}$$

The velocity autocorrelation function is, for $t_2\geq t_1$,

$$\{\langle u(t_2)u(t_1)\rangle\}$$
$$=\exp[-\int_0^{t_2}\beta(s)\,ds]\exp[-\int_0^{t_1}\beta(s)\,ds]\{u^2(0)\}+(k_BT/m)$$
$$\times\{\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]-\exp[-\int_0^{t_2}\beta(s')\,ds']$$
$$\times\exp[-\int_0^{t_1}\beta(s'')\,ds'']\}$$
$$=(k_BT/m)\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]. \tag{25}$$

This result is proved in Appendix B.

Proceeding as in (7)—(14) leads to the conditioned two-point distribution

$$P_2(u_1t_1;u_2t_2)=(2\pi(k_BT/m)\{1-\exp[-2\int_{t_1}^{t_2}\beta(s)\,ds]\})^{-1/2}$$
$$\times\exp\left(-\frac{m(u_2-\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]u_1)^2}{2k_BT(1-\exp[-2\int_{t_1}^{t_2}\beta(s)\,ds])}\right). \tag{26}$$

Associated with this P_2 is the partial differential equation

$$\frac{\partial}{\partial t_2}P_2(u_1t_1;u_2t_2)=\beta(t_2)\frac{\partial}{\partial u_2}(u_2P_2(u_1t_1;u_2t_2))$$
$$+\frac{k_BT}{m}\beta(t_2)\frac{\partial^2}{\partial u_2^2}P_2(u_1t_1;u_2t_2) \tag{27}$$

with the initial equation $P_2(u_1t_1;u_2t_1)=\delta(u_2-u_1)$. Now, again we make the substitutions $t_1=0$, $t_2=t$, $u_1=u(0)$, and $u_2=u$. Then (27) looks like

$$\frac{\partial}{\partial t}P_2(u(0);ut)$$
$$=\beta(t)\frac{\partial}{\partial u}(uP_2(u(0);ut))$$
$$+\frac{k_BT}{m}\beta(t)\frac{\partial^2}{\partial u^2}P_2(u(0);ut) \tag{28}$$

with initial condition $P_2(u(0);u0)=\delta(u-u(0))$. The big difference between these results and those in (20) and (21) is

$$-\dot{\chi}(t_2-t_1)/\chi(t_2-t_1)\rightarrow\beta(t_2). \tag{29}$$

The explicit t_1 dependence of (20) disappears in (27). Moreover, we can always solve (28) starting at any time t_1 with $P_2(u_1;ut_1)=\delta(u-u_1)$, and (28) then yields precisely (26). Therefore, (28) *with appropriate initial conditions* yields (26) for *any* time interval. This is a major characteristic of a bona fide Fokker—Planck equation. However, Eq. (21) does not behave this way. If the interval from t_1 to t_2 is considered and $P_2(u_1;ut_1)=\delta(u-u_1)$ is assumed, then (21) yields

$$P_2(u_1t_1;u_2t_2)$$
$$=\left[2\pi\frac{k_BT}{m}\left(1-\frac{\chi^2(t_2)}{\chi^2(t_1)}\right)\right]^{-1/2}$$
$$\times\exp\left(-\frac{m(u_2-[\chi(t_2)/\chi(t_1)]u_1)^2}{2k_BT(1-\chi^2(t_2)/\chi^2(t_1))}\right). \tag{30}$$

This is *not* equal to (14) except when $\chi(t_2-t_1)=\chi(t_2)/\chi(t_1)$ which only holds when (18) is true, that is, only in the Markovian situation.

V. DISCUSSION OF RESULTS

In this paper we have developed the explicit solution for the generalized Langevin equation using the distribution functions which all depend upon a single, two-time correlation function. In the earlier work of Adelman[1] and Fox[2] the approach was based upon obtaining the Fokker—Planck equation representation. In fact both Adelman and Fox only obtained the Fokker—Planck-like equation identical with (21). They did not obtain (20) which would have made it clear that these equations require a very special initial condition tailored to the specific time interval between t_1 and t_2.

Adelman does suggest that the non-Markovian Langevin description, as in (1), is more fundamental than the Fokker—Planck-like equation description, as in (21). However, after obtaining (21) Adelman overlooks the fact that the Fokker—Planck-like equation only generates the P_2 function for the time interval from 0 to t, and for *no* other interval. He also does not obtain the complete description for arbitrary intervals which is exhibited in (14) and (20). Fox compounds this confusion by noting that the solutions to (21), which he mistakenly takes to be valid for *arbitrary* intervals, generates (30) for arbitrary intervals. While (30) clearly does not describe the actual process given by (1) and (4), as has been pointed out above, it does, *unfortunately*, satisfy identically the Chapman—Kolmogorov—Smoluchowski identity (16) *and* the Doob identity, which in that case is simply

$$\frac{\chi(t_3)}{\chi(t_2)}\frac{\chi(t_2)}{\chi(t_1)}=\frac{\chi(t_3)}{\chi(t_1)}. \tag{31}$$

This "verifies" the *Markov* property! Thus, it appears that the process is really *nonstationary*, *Gaussian*, *Markovian*.

The remarkable feature, which is valid in the Gaussian case anyway, is that the description of all the statistics for the generalized Langevin equation depends on only a single, two-point correlation, $\chi(t_2-t_1)$. Markov processes are always determined completely by a single two-point distribution, P_2. Gaussian, *non*-Markovian processes are *also* completely determined by

$\chi(t_2 - t_1)$, when there is a fluctuation—dissipation relation as in (2). Therefore, nothing is really lost by using a non-Markovian description in place of a Markov description as long as it is Gaussian!

None of the preceding considerations are substantially altered in the multicomponent generalization of (1). One still gets a *stationary, Gaussian, non-Markovian* process which is determined completely by a single, two-point correlation matrix.

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APPENDIX A: PROOF OF (8)

$$\langle[\int_0^{t_2}\chi(t_2-s_2)\tilde{f}(s_2)\,ds_2][\int_0^{t_1}\chi(t_1-s_1)\tilde{f}(s_1)\,ds_1]\rangle$$
$$=\int_0^{t_2}ds_2\int_0^{t_1}ds_1\,\chi(t_2-s_2)\chi(t_1-s_1)k_B T m\beta(|s_2-s_1|) \tag{A1}$$

follows from (2).

The double Laplace transform of the right-hand side is

$$\int_0^\infty dt_2\exp(-zt_2)\int_0^\infty dt_1\exp(-z't_1)\int_0^{t_2}ds_2\int_0^{t_1}ds_1$$
$$\times\chi(t_2-s_2)\chi(t_1-s_1)k_BTm\beta(|s_2-s_1|)$$
$$=k_BTm\int_0^\infty ds_2\int_{s_2}^\infty dt_2\int_0^\infty ds_1\int_{s_1}^\infty dt_1$$
$$\times\exp[-z(t_2-s_2)]\chi(t_2-s_2)$$
$$\times\exp[-z'(t_1-s_1)]\chi(t_1-s_1)\exp(-zs_2)$$
$$\times\exp(-z's_1)\beta(|s_2-s_1|)$$
$$=k_BTm\int_0^\infty ds_2\int_0^\infty d\tau_2\int_0^\infty ds_1\int_0^\infty d\tau_1$$
$$\times\exp(-z\tau_2)\chi(\tau_2)\exp(-z'\tau_1)$$
$$\times\chi(\tau_1)\exp(-zs_2)\exp(-z's_1)\beta(|s_2-s_1|)$$
$$=k_BTm\hat{\chi}(z)\hat{\chi}(z')\int_0^\infty ds_2\int_0^\infty ds_1\exp(-zs_2)$$
$$\times\exp(-z's_1)\beta(|s_2-s_1|). \tag{A2}$$

Now,

$$\int_0^\infty ds_2\int_0^\infty ds_1\exp(-zs_2)\exp(-z's_1)\beta(|s_2-s_1|)$$
$$=\int_0^\infty ds_2\int_0^\infty ds_1\exp[-z(s_2-s_1)]\beta(|s_2-s_1|)$$
$$\times\exp[-(z+z')s_1]$$
$$=\int_0^\infty ds_1\int_{-s_1}^\infty d\sigma\exp(-z\sigma)\beta(|\sigma|)\exp[-(z+z')s_1]$$
$$=\int_0^\infty ds_1\Big(\hat{\beta}(z)+\int_{-s_1}^0 d\sigma\exp(-z\sigma)\beta(|\sigma|)\Big)$$
$$\times\exp[-(z+z')s_1]$$
$$=\frac{\hat{\beta}(z)}{z+z'}+\int_0^\infty ds_1\Big(-\frac{1}{z+z'}\frac{d}{ds_1}$$
$$\times\exp[-(z+z')s_1]\Big)\int_{-s_1}^0\exp(-z\sigma)\beta(|\sigma|)\,d\sigma$$
$$=\frac{\hat{\beta}(z)}{z+z'}+\int_0^\infty ds_1\frac{\exp[-(z+z')s_1]}{z+z'}\exp(zs_1)\beta(s_1)$$
$$=\frac{\hat{\beta}(z)+\hat{\beta}(z')}{z+z'}. \tag{A3}$$

Therefore, we get the identity

$$\Big\langle\Big[\int_0^{t_2}\chi(t_2-s_2)\tilde{f}(s_2)\,ds_2\Big]\Big[\int_0^{t_1}\chi(t_1-s_1)\tilde{f}(s_1)\,ds_1\Big]\Big\rangle$$
$$=k_BTm\hat{\chi}(z)\hat{\chi}(z')\frac{\hat{\beta}(z)+\hat{\beta}(z')}{z+z'}. \tag{A4}$$

Using the definition of $\hat{\chi}(z)$ in (5) gives

$$\hat{\chi}(z)\hat{\beta}(z)=1-z\hat{\chi}(z)\quad\text{and}\quad\hat{\chi}(z')\hat{\beta}(z')=1-z'\hat{\chi}(z'). \tag{A5}$$

These two identities yield

$$k_BTm\hat{\chi}(z)\hat{\chi}(z')\frac{\hat{\beta}(z)+\hat{\beta}(z')}{z+z'}$$
$$=k_BTm\Big(\frac{\hat{\chi}(z')+\hat{\chi}(z)}{z+z'}-\hat{\chi}(z)\hat{\chi}(z')\Big). \tag{A6}$$

In parallel with the identity in (A3), we conclude that (A6) is the double Laplace transform of

$$k_BTm(\chi(|t_2-t_1|)-\chi(t_2)\chi(t_1))$$

which completes the proof of (8) because $t_2\geq t_1$.

APPENDIX B; PROOF OF (25)

It will suffice to verify, for $t_2\geq t_1$, that

$$\langle\{\int_0^{t_2}ds_2\exp[-\int_{s_2}^{t_2}\beta(s')\,ds']\tilde{f}(s_2)/m\}$$
$$\times\{\int_0^{t_1}ds_1\exp[-\int_{s_1}^{t_1}\beta(s')\,ds']\tilde{f}(s_1)/m\}\rangle$$
$$=k_BTm[\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]-\exp[-\int_0^{t_2}\beta(s)\,ds)]$$
$$\times\exp[-\int_0^{t_1}\beta(s')\,ds']]. \tag{B1}$$

From (23) we get

$$\langle\{\int_0^{t_2}ds_2\exp[-\int_{s_2}^{t_2}\beta(s')\,ds']\tilde{f}(s_2)/m\}$$
$$\times\{\int_0^{t_1}ds_1\exp[-\int_{s_1}^{t_1}\beta(s')\,ds']\tilde{f}(s_1)/m\}\rangle$$
$$=2(k_BT/m)\int_0^{t_2}ds_2\int_0^{t_1}ds_1\exp[-\int_{s_2}^{t_2}\beta(s')\,ds']$$
$$\times\exp[-\int_{s_1}^{t_1}\beta(s)ds]\beta(s_2)\delta(s_2-s_1)$$
$$=2(k_BT/m)\exp[-\int_{t_1}^{t_2}\beta(s)ds]\int_0^{t_1}ds_2\int_0^{t_1}ds_1$$
$$\times\exp[-\int_{s_2}^{t_1}\beta(s')\,ds']$$
$$\times\exp[-\int_{s_1}^{t_1}\beta(s)\,ds]\beta(s_2)\delta(s_2-s_1)$$
$$=2(k_BT/m)\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]\int_0^{t_1}ds_2\beta(s_2)$$
$$\times\exp[-2\int_{s_2}^{t_1}\beta(s)\,ds]$$

$$= 2(k_B T/m)\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]\int_0^{t_1} ds_2 \tfrac{1}{2}(d/ds_2)$$

$$\times\exp[-2\int_{s_2}^{t_1}\beta(s)\,ds]$$

$$= 2(k_B T/m)\exp[-\int_{t_1}^{t_2}\beta(s)\,ds]\{\tfrac{1}{2}-\tfrac{1}{2}\exp[-2\int_0^{t_1}\beta(s)\,ds]\}$$

$$= (k_B T/m)[\exp\int_{t_1}^{t_2}\beta(s)\,ds] - \exp[\int_0^{t_2}\beta(s)\,ds]$$

$$\times\exp[-\int_0^{t_1}\beta(s')\,ds']], \tag{B2}$$

which completes the proof.

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