

Quantum chaos in a two-level system in a semiclassical radiation field

John Eidson and Ronald F. Fox

School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

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The semiclassical Bloch-Maxwell system is studied for chaos. Diagnostic criteria for chaos such as continuous power spectra and rapidly decaying correlations are shown not to be definitive. They may occur simultaneously with a vanishing maximum Liapunov exponent. The presence of a feedback effect on the radiation field is shown to lead to the possibility of *bona fide* chaos with a continuous spectrum and a positive Liapunov exponent. Whether or not chaos is exhibited in this system depends upon initial conditions as well, and cannot be deduced from the Hamiltonian alone.

Chaos is a well-established property of classical dynamical systems. To what extent an analogous behavior occurs in quantal systems is still being investigated.¹⁻⁵ In this paper we discuss a two-level quantal system driven by a semiclassical electromagnetic field. This description is intermediate between a purely classical and a purely quantal treatment. We are able to clearly delimit the boundary between chaotic and nonchaotic dynamics. In addition the mechanism responsible for chaos in our system is elucidated and is suggestive of *bona fide* chaos in a related, purely quantal system.

The dynamics to be described below involves a finite system of coupled first-order differential equations which exhibit a bounded phase flow. Chaos refers to the existence of a positive Liapunov exponent for this flow.⁶ Power spectra and correlation function decays^{3,7} may be used as a diagnostics for chaos but are not definitive evidence.

The system we consider is in essence the Bloch-Maxwell system for a two-level system in an electromagnetic field. The Hamiltonian for this system is

$$H = \frac{1}{2} \hbar \omega_0 \sigma_z + \hbar \lambda A(t) \sigma_x, \tag{1}$$

in which σ_x and σ_z are Pauli operators for the two-level system, $\hbar \omega_0$ is the level spacing, and $\lambda A(t)$ is the semiclassical electromagnetic field. The Heisenberg equations of motion for the Pauli operators are

$$\dot{\sigma}_x = -\omega_0 \sigma_y, \tag{2a}$$

$$\dot{\sigma}_y = \omega_0 \sigma_x - 2\lambda A(t) \sigma_z, \tag{2b}$$

$$\dot{\sigma}_z = 2\lambda A(t) \sigma_y. \tag{2c}$$

The expectation values with respect to the initial state of the two-level system satisfy identical equations because the operator equations in (2a)–(2c) are linear in the Pauli operators. Thus, we get a system of three coupled, ordinary, first-order differential equations

$$\dot{x} = -\omega_0 y, \tag{3a}$$

$$\dot{y} = \omega_0 x - 2\lambda A(t) z, \tag{3b}$$

$$\dot{z} = 2\lambda A(t) y, \tag{3c}$$

in which $x = E x(\sigma_x)$, etc. So far, these equations constitute the Bloch equations and are nonautonomous because of $A(t)$. For $A(t)$ we consider two cases: (1) $A(t)$ satisfies the Maxwell equation^{1,4}

$$\ddot{A} + \omega^2 A = -2N\lambda\omega x \tag{4}$$

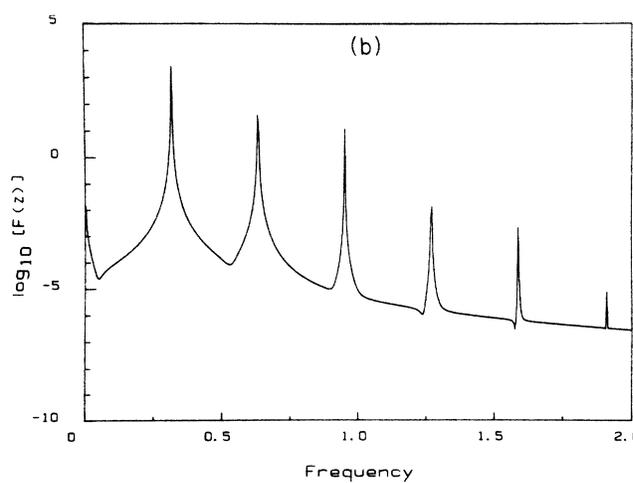
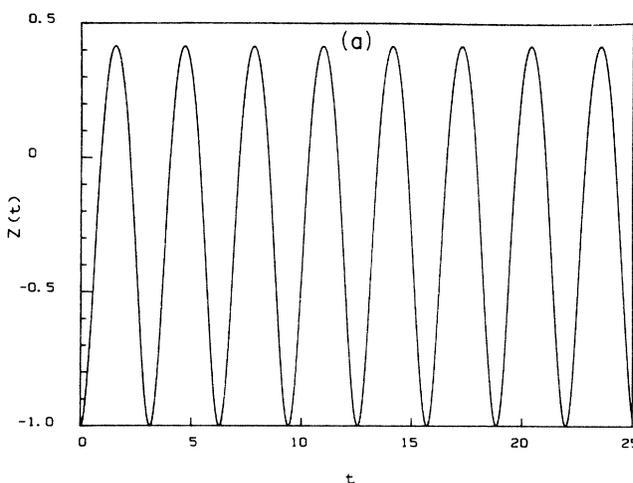


FIG. 1. (a) z trajectory for Eqs. (3a)–(3c); $N = \omega_0 = 0$, $A(t) = \cos(t)$; $\lambda = 1$, $x(0) = y(0) = 0$, $z(0) = -1$. (b) Fast Fourier transform (FFT) of the z trajectory, $F(z)$.

and (2) $A(t)$ is prescribed to be given by

$$A(t) = \cos(\omega t) \cos(\omega' t), \quad (5)$$

in which ω and ω' are incommensurate.⁵ This second case was recently discussed by Pomeau *et al.*⁵ and was shown to exhibit a continuous power spectrum as well as a rapidly decaying correlation even though it does not possess a

positive Liapunov exponent. The first case, on the other hand, does possess a positive Liapunov exponent for appropriately chosen initial conditions, as well as nonchaotic dynamics for other initial conditions. These behaviors are a consequence of the simple, linear feedback term, x , in Eq. (4).

We will describe the behavior of both cases in three distinct situations: (1) $\omega_0=0$ (and $N=0$ for case 1); (2)

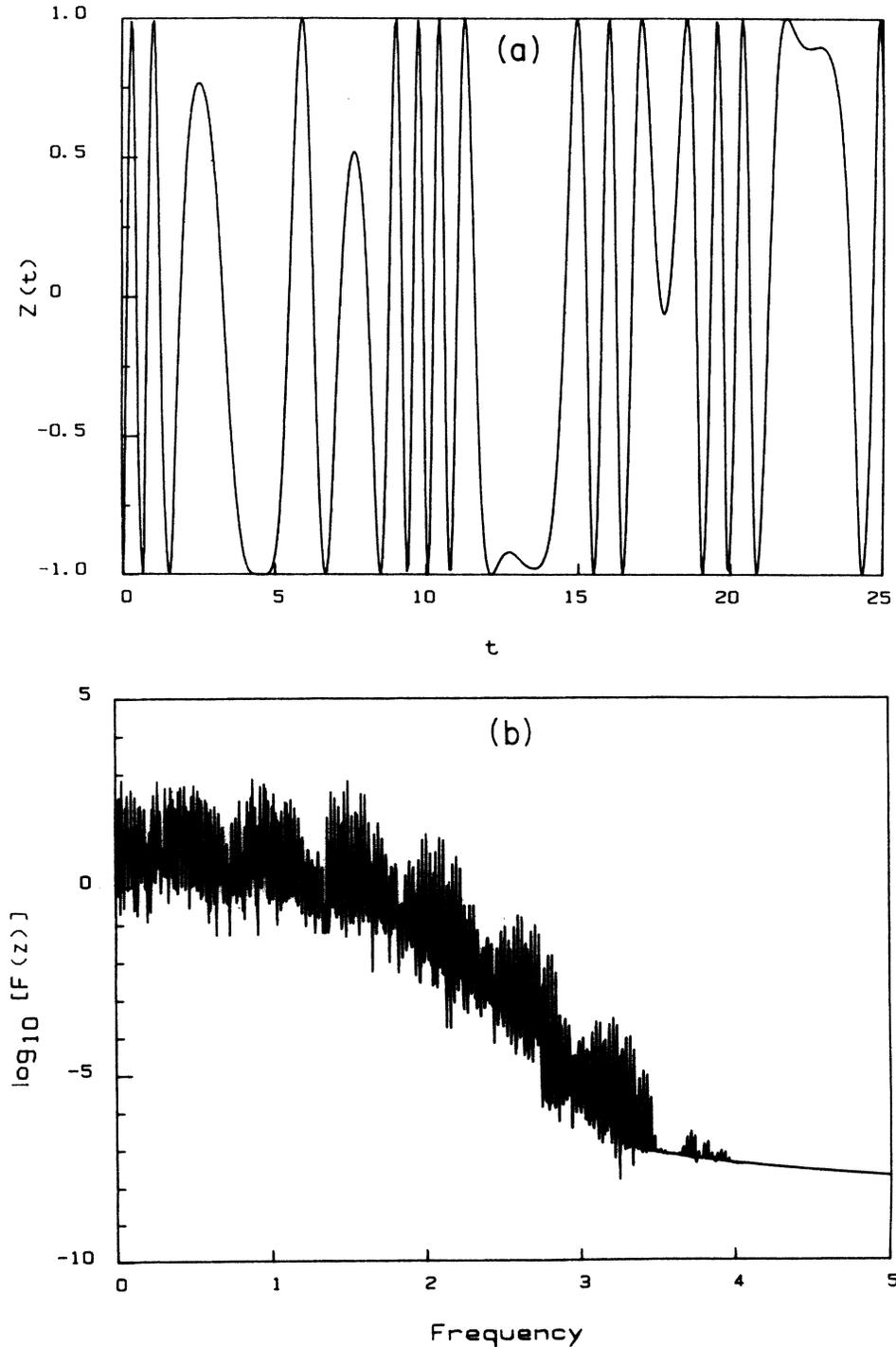


FIG. 2. (a) z trajectory, $\omega_0=0$, $A(t)=\cos(\omega t)\cos(\omega' t)$, $\omega=17\,711/28\,657$, $\omega'=4637/13\,313$, $\lambda=5$, same initial conditions as in Fig. 1. (b) FFT of the z trajectory, $F(z)$. (c) Expanded scale for b .

$\omega_0 \neq 0$ ($N=0$ for case 1); and (3) $\omega_0 \neq 0$ ($N=1$ for case 1). In situation (1) both cases yield explicit integrals for the solutions. The power spectra show well-defined peaks in case (1) but an apparently continuous spectrum in case (2), and the Liapunov exponents are zero. In situation (2), both cases may be described by time-ordered exponential solutions which cannot be rendered in any simpler form. In case (1), the power spectrum still shows well-defined peaks and the Liapunov exponent vanishes. In case (2) the power spectrum appears continuous, a correlation shows rapid decay, but the Liapunov exponent is zero. In situation (3), case (1) exhibits chaos, i.e., a positive Liapunov exponent, for appropriate initial conditions. Case (2), on the other hand, is the same as in situation (2). Greater detail for each situation appears below.

Situation (1): $\omega_0=0$ and $N=0$. This situation reduces the problem to two variables, y and z . They satisfy the equation

$$\frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = 2\lambda A(t) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} \tag{6}$$

which has the general solution

$$\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \cos \left[2\lambda \int_0^t ds A(s) \right] & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ + \sin \left[2\lambda \int_0^t ds A(s) \right] & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} y(0) \\ z(0) \end{bmatrix}. \tag{7}$$

In case (1), Eq. (4) becomes

$$\ddot{A} + \omega^2 A = 0 \tag{8}$$

which has the solution

$$A(t) = A(0)\cos(\omega t) + \dot{A}(0)\sin(\omega t). \tag{9}$$

This expression is easily inserted into solution (7) and the integrals are easily executed. In case (2), $A(t)$ is given by (5), and insertion into solution (7) again leads to easily executed integrals. Thus, in each case we have explicit integral solutions. In Figs. 1 and 2, we show plots of $z(t)$ and its corresponding power spectrum for each case. The Liapunov exponent in each case is zero, as can be deduced analytically as well as numerically. In case (2) the power spectrum appears continuous. For $A(t)$ given by (5) with incommensurate ω and ω' , the nonlinear combinations created by the solution (7) create harmonic mixtures and overtones in an apparently continuous manner.

Situation (2): $\omega_0 \neq 0$ and $N=0$. Now we are back to a three-variable problem. However, for case (1) the x -feedback term has been eliminated and $A(t)$ is again given by Eq. (9). In essence this makes case (1) equivalent to a special case treated by Pomeau *et al.*⁵ in which the prescribed electromagnetic field is given by a pure trigonometric function instead of by a product of two such functions as in Eq. (5). In each case we know $A(t)$ explicitly and can express Eqs. (3a)–(3c) in the form

$$\frac{d}{dt} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & -2\lambda A(t) \\ 0 & 2\lambda A(t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{10}$$

Because the coupling matrix does not commute with itself at unequal times, the solution to this equation must be expressed in terms of a time-ordered exponential.⁸

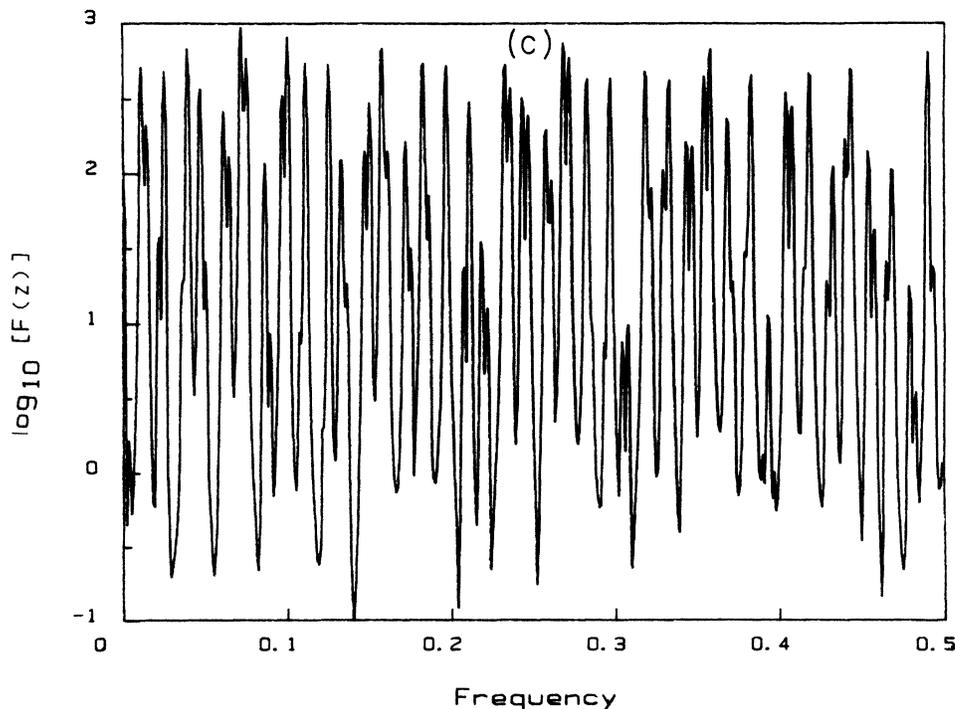


FIG. 2. (Continued).

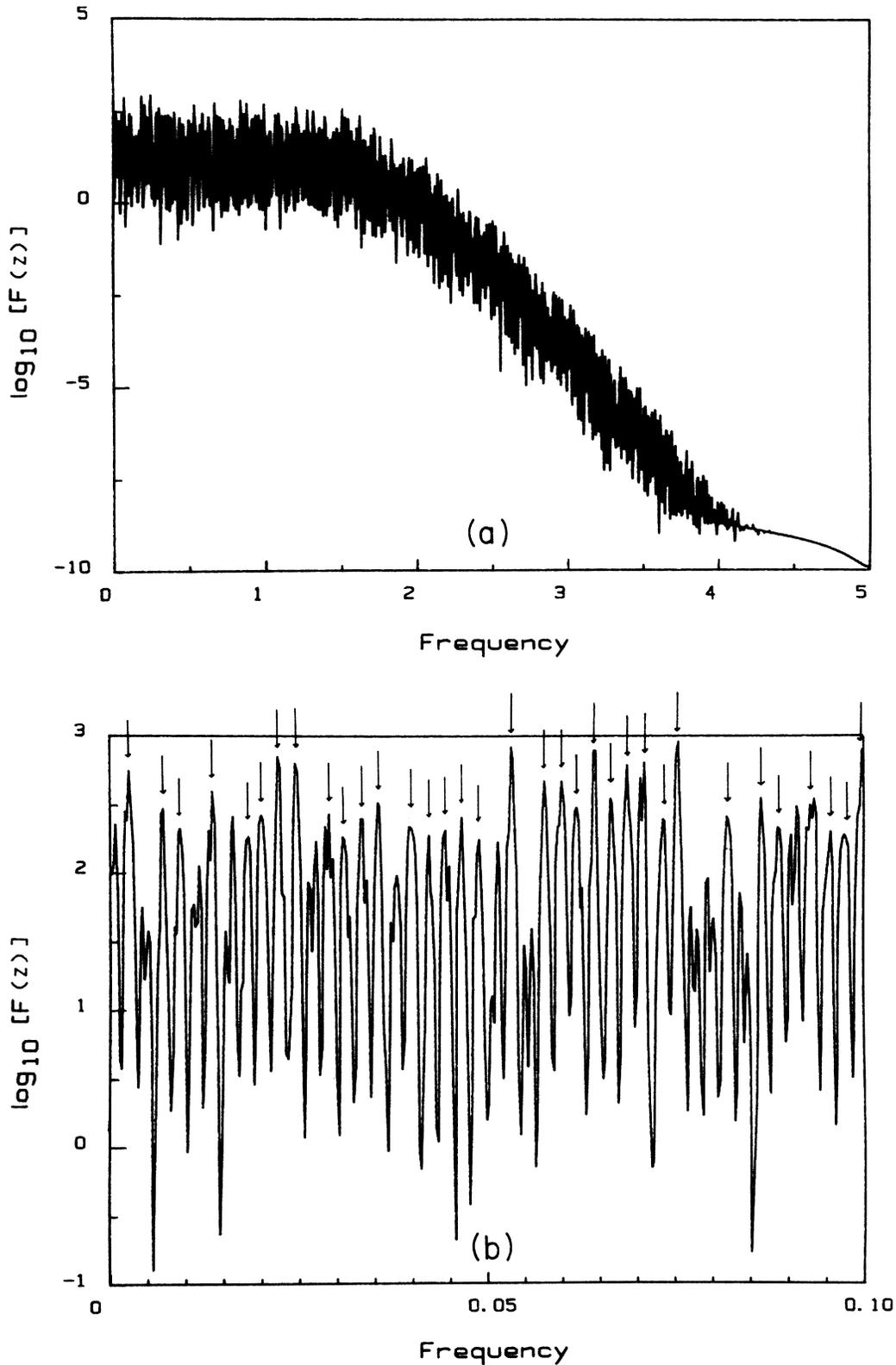


FIG. 3. (a) FFT for the z trajectory for Eqs. (3a)–(3c), $\omega_0=1$, $A(t)=\cos(\omega t)\cos(\omega' t)$, $\omega=34/55$, $\omega'=34/89$, $\lambda=5$, same initial conditions as Fig. 1. (b) Expanded scale for a . The peaks identical by arrows are simple combinations of ω and ω' , $(M\omega - N\omega')/\pi$, to within ± 0.0005 . From left to right, the values for (M, N) for each peak are: $(-21, -34)$, $(-8, -13)$, $(26, 42)$, $(-16, -26)$, $(-3, -5)$, $(-24, -39)$, $(10, 16)$, $(-11, -18)$, $(2, 3)$, $(-19, -31)$, $(15, 24)$, $(-6, -10)$, $(7, 11)$, $(-14, -23)$, $(20, 32)$, $(-1, -2)$, $(-22, -36)$, $(-9, -15)$, $(4, 6)$, $(-17, -28)$, $(17, 27)$, $(-4, -7)$, $(-25, -41)$, $(9, 14)$, $(-12, -20)$, $(22, 3)$, $(1, 1)$, $(-7, -12)$, $(6, 9)$, $(-15, -25)$, $(-2, -4)$, $(-23, -3)$, $(11, 17)$, $(-10, -17)$.

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = T \exp \left[\int_0^t ds \begin{pmatrix} 0 & -\omega_0 & 0 \\ \omega_0 & 0 & -2\lambda A(s) \\ 0 & 2\lambda A(s) & 0 \end{pmatrix} \right] \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}. \quad (11)$$

While this expression cannot be rendered in a simple, explicit, reduced form, it does not introduce sufficient complexity to create chaos. In case (1), representative figures for $z(t)$ and its power spectrum are given by Figs. 1(a) and 1(b) of Pomeau *et al.*⁵ Well-defined peaks are seen in the power spectrum. On the other hand, for case (2) Figs. 2(a) and 2(b) of Pomeau *et al.*⁵ provide $z(t)$ and its power spectrum. Now, a continuous spectrum is again seen; nevertheless, the Liapunov exponent in each case is zero. Once again this may be demonstrated analytically because Eq. (10) is linear and $A(t)$ is independent of the initial values for x , y , and z . Consequently, not only is $x^2 + y^2 + z^2$ conserved, but $(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is also conserved, that is, the norm of any tangent vector to the flow is conserved. The numerical algorithm⁹ for computing the Liapunov exponent will automatically yield zero in such a situation.

We have studied in detail the origin of the apparently continuous spectrum. Pomeau *et al.* used $\omega = 17711/28657$ and $\omega' = 4637/13313$ in their numerical computations. We have instead looked at a succession of Fibonacci series ratios which exhibit the same behavior as seen by Pomeau *et al.* when these ratios are comparable to the values quoted above. Specifically, if 1,1,2,3,5,8,13,21,... are the Fibonacci numbers $F_1, F_2, F_3, F_4, F_5, \dots$, respectively, then we choose $\omega = F_n/F_{n+1}$ and $\omega' = F_n/F_{n+2}$. Thus, $F_{22}/F_{23} = 17711/28657$. However, an apparently continuous spectrum can be obtained for $\omega = F_9/F_{10} = 34/55$ and $\omega' = F_9/F_{11} = 34/89$, as is shown in Fig. 3(a). We have enlarged the initial portion of this spectrum in Fig. 3(b) and have identified many of the peaks as simple combinations of ω and ω' . The combination $34\omega' - 21\omega$, i.e., $F_9\omega' - F_8\omega$ is particularly small (0.0069) and its mixture with other combinations as well as its overtones produces the apparently continuous character to the spectrum. As the Fibonacci index n increases, a corresponding combination with the value $\Omega_n = F_n F_n / F_{n+2} - F_{n-1} F_n / F_{n+1}$ appears and is even smaller, e.g., for $n = 20$ this difference is now 0.000035. This value makes it a practical impossibility to enlarge the spectrum enough to be able to see and identify the discrete peaks which we have shown for $n = 9$. As $n \rightarrow \infty$, $\Omega_n \rightarrow 0$ and the spectrum becomes truly continuous, corresponding to a quasiperiodic trajectory.

Situation (3): $\omega_0 \neq 0$ and $N = 1$. Since case (2) in this

situation is identical with case (2) in the previous situation, we will only discuss case (1). Now it is impossible to solve explicitly for $A(t)$ and one is forced to analyze a five-variable system of coupled first-order equations equivalent to Eqs. (3a)–(3c) and (4):

$$\dot{x} = -\omega_0 y, \quad (12a)$$

$$\dot{y} = \omega_0 x - 2\lambda A(t) z, \quad (12b)$$

$$\dot{z} = 2\lambda A(t) y, \quad (12c)$$

$$\dot{A} = -\omega B, \quad (12d)$$

$$\dot{B} = \omega A + 2\lambda x. \quad (12e)$$

We have studied this system extensively.⁴ For special initial conditions [$x(0) = y(0) = 0$, $z(0) = 1$, $\lambda = 0.5$] a continuous power spectrum^{2,4} for $z(t)$ is found as well as a positive Liapunov exponent^{1,2,4} (0.196). It is also possible to choose a different set of initial conditions [$x(0) = z(0) = 0$, $y(0) = 1$, $\lambda = 0.05$] such that nonchaotic dynamics is observed, that is for which the Liapunov exponent vanishes (< 0.003). Our analysis⁴ elucidated the mechanism for this behavior in terms of a periodically perturbed pendulum dynamics which is embedded in the Bloch-Maxwell system. The pendulum dynamics is known to be a generic source of chaos in classical systems, as was shown by Chirikov.¹⁰ This pendulum mechanism has its origin in the simple, linear x -feedback term in Eq. (12e).

The Bloch-Maxwell system with a semiclassical radiation field is on the borderline between chaotic and nonchaotic dynamics. We have shown that diagnostic criteria⁵ for chaos such as a continuous power spectrum or a rapidly decaying correlation are not definitive. Only the positivity of the maximum Liapunov exponent confirms *bona fide* chaos in a bounded, first-order differential, phase flow. In the Bloch-Maxwell system, chaos is possible only if there is feedback on the radiation field. Otherwise the system is at most ergodic. Even when chaos is possible for certain initial conditions, it is also possible to exhibit nonchaotic dynamics for different initial conditions. Therefore, criteria based solely upon the structure of the Hamiltonian or its eigenspectrum cannot be correct. The Bloch-Maxwell system appears to be the minimal system in which each of these distinctions may be demonstrated.

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¹P. I. Belobrov, G. M. Zaslavskii, and G. Kh. Tartakovskii, Zh. Eksp. Teor. Fiz. 71, 1799 (1976) [Sov. Phys.—JETP 44, 945 (1977)].

²P. W. Milonni, J. R. Ackerhalt, and H. W. Galbraith, Phys. Rev. Lett. 50, 966 (1983).

³G. Casati and I. Guarneri, Phys. Rev. Lett. 50, 640 (1983).

⁴R. F. Fox and J. Eidson, Phys. Rev. A 34, 482 (1986).

⁵Y. Pomeau, B. Dorizzi, and B. Grammaticos, Phys. Rev. Lett. 56, 681 (1986).

⁶C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke, Physica 13D, 261 (1984).

⁷M. Shapiro and G. Goelman, Phys. Rev. Lett. 53, 1714 (1984).

⁸R. F. Fox, Phys. Rep. 48, 179 (1978), see p. 238.

⁹G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A 14, 2338 (1976); G. Benettin and L. Galgani, J. Stat. Phys. 27, 153 (1982).

¹⁰B. V. Chirikov, Phys. Rep. 52, 265 (1979).