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Amplification of intrinsic fluctuations by chaotic dynamics in physical systems

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A quantitative method for the treatment of large-scale intrinsic fluctuations amplified by chaotic trajectories in macrovariable physical systems is presented. Paradigmatic results for the Rossler model and preliminary computational results for chaotic Josephson junctions and for chaotic multimode Nd:YAG (yttrium aluminum garnet) lasers are described. These studies are directed towards identification of a real physical system in which experimental confirmation may be realized. The probability distribution on the intrinsic-noise-modified, chaotic attractor is identified as a likely candidate for comparison of experiment and theory.

I. INTRODUCTION

In several recent papers,[1–4] we showed that chaotic dynamics can cause macroscopic growth of intrinsic fluctuations in a macrovariable system. Implications of this effect were suggested for systems as diverse as chemical, hydrodynamic, electronic, and quantum. In this paper, we propose a highly accurate approach to the theoretical description of such large-scale fluctuations. Our proposal is based upon a limit theorem for Markov chains proved by Kurtz[5,6] in 1975, long before its relevance for chaotic dynamics could be appreciated.

That chaotic dynamics and the growth of intrinsic fluctuations are related to each other is a consequence of each being fundamentally tied to a dynamical quantity called the Jacobi matrix.[2] A quantitative characterization of chaos is provided by the largest Liapunov exponent, which when positive, implies chaos.[7] The computation of the largest Liapunov exponent directly utilizes the instantaneous values of the Jacobi matrix.[8] Similarly, the growth of the intrinsic fluctuations is made quantitative by following the time evolution of the covariance matrix.[9,10] Again, the computation of the covariance matrix evolution directly utilizes the instantaneous values of the Jacobi matrix.[2] This dual role of the Jacobi matrix and the consequence that intrinsic fluctuations become very large in a chaotically dynamic system was apparently noticed for the first time only recently.[1,2,4]

In order to make this connection explicit, imagine a macrovariable system described by N macrovariables $M_i(t)$ for $i=1,2,\ldots,N$ satisfying N coupled, nonlinear, ordinary, differential equations

$$\frac{d}{dt}M_i(t)=F_i(\mathbf{M}(t)) \tag{1}$$

in which the F_i's are N, generally nonlinear functions of the M_k's. The Jacobi matrix $J_{ik}(t)$ is defined[7] by

$$J_{ik}(t)=\frac{\partial F_i}{\partial M_k(t)} \tag{2}$$

for each instant of time. It has been shown that the largest Liapunov exponent for this dynamics λ is given by[8]

$$\lambda=\underset{n\to\infty}{\text{limit}}\frac{1}{2n}\ln\{\mathrm{Tr}[\underline{J}^{\dagger}(n)\underline{J}(n)]\} \tag{3}$$

in which $\underline{J}^{\dagger}(n)$ is the adjoint of $\underline{J}(n)$. On the other hand, it has also been established that if Eq. (1) is the macroscopic limit of an embedding (see below) master equation (i.e., some "largeness parameter," say Ω, is allowed to go to an infinite limit), and if the scaled linearized deviations from the deterministic solutions to Eq. (1) are denoted by $\mu_i(t)=\Omega^{1/2}\Delta M_i(t)$ [where $\Delta M_i(t)$ is the unscaled deviation], and if the covariance matrix for these deviations (fluctuations) is denoted by $C_{ik}(t)=\langle\mu_i(t)\mu_k(t)\rangle$, where $\langle\;\rangle$ denotes averaging with respect to the master equation's probability distribution, then $C_{ik}(t)$ satisfies

$$\frac{d}{dt}C_{ik}(t)=J_{ij}(t)C_{jk}(t)+C_{ij}(t)J_{kj}(t)+R_{ik}(t)\;, \tag{4}$$

in which $R_{ik}(t)$ is explicitly determined from the master equation. The exponential divergence of fluctuations in the limit of large Ω is reflected in the fact that Eq. (3) is also valid if $\underline{C}^{\dagger}(t)$ and $\underline{C}(t)$ are substituted in place of $\underline{J}^{\dagger}(n)$ and $\underline{J}(n)$ on the right-hand side.[1,4]

The covariance matrix evolution equation involves a linearization of the macrovariable dynamics instantaneously in time. This, of course, produces the Jacobi matrix dependence, but it also means that once the fluctuations have grown even a little bit, the linearized equations lose their validity. In our earlier work[1,2,4] we stressed this point, and noted that while the covariance matrix evolution permitted computation of the largest Liapunov exponent, it did not accurately describe the fluctuations once they grew to macroscopic size. In order to obtain the large-scale fluctuations, a mesoscopic underpinning of the macrovariable equations is required.[2,3] One way to accomplish this is to embed the macrovariable equations in a mesoscopic master equation and deduce the time evolution of the underlying probability distribution.

Given the master equation, which is not generally agreed upon for all interesting contexts (e.g., hydrodynamics[9]), one must solve it, albeit numerically. This last task is formidable for multivariable systems and has prompted us to look for alternative approaches.

One alternative to solving the master equation for the probability distribution is to implement the process described by the master equation as a stochastic process.[9] This requires performing many realizations of the stochastic process in order to build up the equivalent probability distribution. A theorem due to Kurtz[5,6] is closely related to this approach and establishes a highly accurate approximation to the stochastic process needed. Because the implementation of Kurtz's theorem for this purpose looks very much like merely adding *extrinsic* fluctuations to the macrovariable equations, we will attempt to distinguish clearly the important differences.

Kurtz's theorem may be implemented either as a stochastic process or as an equivalent Fokker-Planck equation. In the latter guise, it is a so-called nonlinear Fokker-Planck equation that is used.[11] In other contexts, objections to such an equation have been voiced.[12] The chief objection is that the averaged quantities determined by a nonlinear Fokker-Planck equation do not satisfy the macrovariable equations because averages of nonlinear expressions are not equal to identical nonlinear expressions of the averages. However, this is precisely the circumstance that is relevant when intrinsic fluctuations grow to large scale. Thus we again find it necessary to contrast what is done here with earlier applications of some closely related methods. Context will prove to be the crucial distinguishing element.

The remainder of this paper is divided into three sections. In Sec. II, we define a variety of kinds of noise or fluctuations. We do this because earlier work[13] does not distinguish the many types of noise discussed here and the same words we use are used with different meaning in these earlier papers. In Sec. III, we discuss the transition from a mesoscopic picture to a macrovariable dynamics. Both the traditional view[11] of this transition and Kurtz's theorem[5,6] will be presented. Certain technical matters regarding the application of Kurtz's theorem to our problems will be addressed. In Sec. IV, we conclude the paper with three examples. The Rossler model[14] is used as a paradigm for the description of the growth of fluctuations on a chaotic trajectory. We establish the probability distribution on an attractor as a good candidate for the comparison of experiment and theory. The amplification of intrinsic noise on chaotic trajectories produces a probability distribution noticeably different from the corresponding, noise-free invariant measure. Preliminary results from a detailed theoretical study of fluctuations in a chaotic Josephson junction[15] are presented. Similar results, with the possibility of future experimental confirmation, for a chaotic multimode Nd:YAG (yttrium aluminum garnet) laser[16] are outlined. These examples provide insight into the methods and their consequences.

II. NOISES

In order to minimize misunderstanding, we will distinguish among several distinct types of noise.[13,17] In the vast literature covering noise in physical systems, words such as *noise, fluctuations,* and *random* have been applied to processes of rather different origin. In some cases, the established usage is so ingrained that alternative usage is easily misconstrued. To define our usage here as clearly as possible, five classes of noise are distinguished: (i) instrumental, (ii) initial data, (iii) external reservoir, (iv) intrinsic molecular, and (v) deterministic chaos.

Instrumental noise is the systematic noise associated with making observations, either in real experiments or in numerical simulations. It is the noise associated with the limits of resolution in the observation procedure. If, for example, σ is the standard deviation for the limit of resolution, no observation will resolve quantities below the σ scale. At the same time, observations will also be no worse than the scale set by σ. This feature is in marked contrast to what will be seen regarding intrinsic fluctuations below.

Uncertainties in the precision of the initial data introduce another kind of noise. One must consider what happens to an ensemble of initial states, each of which is consistent with the limited precision of the initial data. If the dynamics is dissipative and involves an attractor, then the ensemble of initial data will end up as an ensemble distributed over the attractor. For an ergodic attractor, this final ensemble will be an invariant distribution quite independent of its initial properties. Therefore, properties of the stationary ergodic attractor really do not depend on the initial data noise.

Identifiable physical systems are isolated from the rest of the world by container boundaries. These container walls are in contact with the rest of the world. In this way, every system is coupled to a heat bath, or a pressure reservoir, etc. This introduces another kind of noise that we will call *extrinsic* noise. It is essentially independent of the nature of the system, depending instead on how the system is isolated from the rest of the world. In mathematical modeling, this type of noise is introduced by simply adding noise terms to the deterministic equations. The noise properties are introduced through various parameters that are fundamentally independent of the system and the system state. Most earlier studies of the interaction of noise and chaos are concerned with this sort of extrinsic noise.[13,17]

The type of noise upon which we focus attention in this paper is *intrinsic* molecular noise. By this expression, we refer to the molecular composition of real physical systems that are otherwise described by macrovariable equations. The macrovariables refer to macroscopic amounts of matter and, therefore, represent some sort of averaging over an underlying microscopic, or perhaps mesoscopic description.[9] Consequently, associated with each macrovariable is an intrinsic fluctuation of molecular origin. Frequently, these fluctuations are ignored and only the macrovariables are studied. However, light scattering[18] from a hydrodynamic system can be accounted for quantitatively only by working out the dynamics of the fluctuations as well as the macrovariables. Near full equilibrium or near a stable steady state, the fluctuations in no way affect the macrovariable dynamics. For chaotic macrovariable dynamics, however, we have shown that the intrinsic fluctuations are amplified to macroscopic size so that the macrovariable description might be markedly modified. The central purpose of this paper is to present a procedure for an accurate quantitative treatment of chaotic dynamics including amplified intrinsic fluctuations. This treatment of chaotic, intrinsic fluctuations does not appear in any of the earlier literature.

The reader should not confuse our object of study, namely, the amplification of *intrinsic* fluctuations by chaotic dynamics, with a prevalent usage in the literature, wherein wild macrovariable trajectories of chaotic dynamics are themselves referred to as "enhanced fluctuations." This latter usage is consistent with the notion of "deterministic randomness" that also has wide currency. These usages ignore intrinsic molecular fluctuations and refer only to the chaotic macrovariable trajectories as noise. In this light, it is significant that recent research[19] has begun to emphasize the ordered structure of chaotic macrovariable trajectories by showing how to systematically approximate them in terms of unstable periodic orbits. This research is shifting the emphasis from "deterministic randomness" to "ordered chaos." Perhaps this shift will help to eliminate confusion between the wild, chaotic macrovariable trajectories and amplification of *intrinsic* noise.

III. MESOSCOPIC TO MACROSCOPIC TRANSITION

The macroscopic description[9,11] of physical systems, e.g., hydrodynamics and chemical reactions, involves macrovariable equations in which the dependent variables refer to quantities representing averages over the properties of many constituent molecules. When intrinsic fluctuations are totally ignored, a deterministic description is obtained, usually in the form of ordinary or partial differential equations with precise initial and/or boundary conditions. Measurements on such systems often involve scattering probes, e.g., light scattering, that necessitate a quantitative treatment[18] of the intrinsic fluctuations since the scattering is determined by fluctuation correlations. This leads to a stochastic adjunct to the macrovariable description.

There are several ways to obtain a quantitative description of the intrinsic fluctuations. For the linear regime near full equilibrium or near a stable steady state, the Onsager theory has been generalized[20] so that the fluctuation equations may be written down directly from the macrovariable equations through imposition of the fluctuation-dissipation relation which connects the strength of the fluctuations to the magnitude of associated dissipative parameters. For example, in hydrodynamics, the magnitude of the velocity field fluctuations is determined by the viscosity. In order to treat the fluctuations in the dynamical regime further away from full equilibrium or a stable steady state, where nonlinearities may be important, it is necessary to go beyond just the fluctuation-dissipation relation and to obtain a fuller treatment of the dynamics of the intrinsic fluctuations.[9] While some special cases have been treated successfully by kinetic theory,[21] a more general approach is that of the master equation.[9,11] This approach is a mesoscopic description that provides the time evolution of the entire probability distribution for the intrinsic fluctuations and subsumes all of their properties including the fluctuation-dissipation relation.

For spatially homogeneous chemical reactions, the master equation approach is well developed.[22] In fact, several quite rigorous limit theorem[23] results, also due to Kurtz, have been obtained in this case. For hydrodynamics,[9,11] however, a generally accepted master equation for all fluid densities does not exist yet, although in the dilute fluid regime, Boltzmann's equation can be thought of as serving the purpose. Therefore, some of what we have to say about master equations can already be realized in certain contexts, whereas in other contexts, the master equation itself is still to be constructed. Nevertheless, after reviewing the properties of the master equation to macrovariable equation transition, we will present a new approach[5,6] to large-scale fluctuations that does not require the master equation description *per se,* even though this alternative is also mesoscopic.

Equation (1) represents a typical macrovariable equation in the form of an ordinary differential equation. Without loss of generality, we will restrict our remarks in this paper to such equations because most partial differential equations can be recast as ordinary differential equations either through expansions in Fourier modes or by discretizing space. In fact, the typical nonlinear partial differential equation must be treated numerically, in which case one or the other of these treatments is required. The objective of the master equation treatment associated with Eq. (1) is twofold. First, the master equation must imply Eq. (1) in the macroscopic limit for which some scaling parameter, say Ω, is made infinitely large.[11] Second, this same limit must yield the equation for the intrinsic fluctuations associated with the macrovariables by the master equation. The proper physical interpretation of these relations is that the fundamental physics is given by the master equation and both the deterministic macrovariable equations and the fluctuation equations are approximate representations of the information contained in the master equation, the approximation being the better as $\Omega \to \infty$.

The form of the general master equation associated with Eq. (1) is[11,22]

$$\frac{\partial}{\partial t}P(\mathbf{m},t)=\int d^N m'[\,W(\mathbf{m},\mathbf{m}')P(\mathbf{m}',t) - W(\mathbf{m}',\mathbf{m})\mathbf{P}(\mathbf{m},\mathrm{t})]\ , \tag{5}$$

in which $P(\mathbf{m},t)$ is the probability density for $\mathbf{M}(t)$ values, i.e., $P(\mathbf{m},t)d\mathbf{m}$ is the probability that the values of $\mathbf{M}(t)$ at time t are between $\mathbf{m}$ and $\mathbf{m}+d\mathbf{m}$, component by component; $W(\mathbf{m},\mathbf{m}')dt$ is the transition probability for $\mathbf{M}(t)$ values to change from $\mathbf{m}'$ to $\mathbf{m}$ in dt; and $W(\mathbf{m},\mathbf{m}')$ is of order Ω for $|\mathbf{m}-\mathbf{m}'|$ of order $1/\Omega$. In the limit $\Omega \to \infty$, we identify the macrovariable as the average

$$\mathbf{M}(t)=\langle \mathbf{m} \rangle = \int d^N m\ \mathbf{m} P(\mathbf{m},t)\ . \tag{6}$$

The transition moments[11] are defined by

$$K_i^{(1)}(\mathbf{m},t)=\int d^N m'(m_i'-m_i)W(\mathbf{m}',\mathbf{m}) \ , \tag{7}$$

$$K_{ij}^{(2)}(\mathbf{m},t)=\int d^N m'(m_i'-m_i)(m_j'-m_j)W(\mathbf{m}',\mathbf{m}) \ , \tag{8}$$

etc. The Ω properties of W imply[23] that $K^{(1)}\approx O(1)$, $K^{(2)}\approx O(1/\Omega)$, and generally $K^{(n)}\approx O(1/\Omega^{n-1})$. Using these transition moments, the master equation may be rewritten in the equivalent Kramers-Moyal form[24,25]

$$\frac{\partial}{\partial t}P(\mathbf{m},t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left[\prod_{j=1}^{n}\frac{\partial}{\partial m_{k_j}}\right] \times[K^{(n)}_{k_1k_2\cdots k_n}(\mathbf{m})P(\mathbf{m},t)] \ . \tag{9}$$

With these properties, the macroscopic limit, i.e., $\Omega\rightarrow\infty$, implies[2]

$$\frac{\partial}{\partial t}P_\infty(\mathbf{m},t)=-\frac{\partial}{\partial m_i}[K_i^{(1)\infty}(\mathbf{m})P_\infty(\mathbf{m},t)] \ , \tag{10}$$

where the repeated indices in both Eq. (9) and (10) imply a summation and where the subscript (superscript) ∞ denotes the macroscopic limit of the corresponding quantity. This partial differential equation is very special since its derivatives are all first order. This means that if the initial values for the $\mathbf{m}$ components are given precisely, i.e., $P_\infty(\mathbf{m},0)=\delta(\mathbf{m}-\mathbf{m}_0)$, then the solution to Eq. (10) is simply[2]

$$P_\infty(\mathbf{m},t)=\delta(\mathbf{m}-\mathbf{m}(t)) \ , \tag{11}$$

where $\mathbf{m}(t)$ satisfies the system of coupled ordinary differential equations

$$\frac{d}{dt}m_i(t)=K_i^{(1)\infty}(\mathbf{m}(t)) \ . \tag{12}$$

Moreover, if we apply the averaging defined in Eq. (16), we obtain the equations

$$\frac{d}{dt}M_i(t)=\langle K_i^{(1)\infty}(\mathbf{m})\rangle=K_i^{(1)\infty}(\langle\mathbf{m}\rangle) =K_i^{(1)\infty}(\mathbf{M}(t)) \tag{13}$$

on account of the Dirac δ-function solution (11). Thus $\mathbf{M}(t)$ is the same as $\mathbf{m}(t)$, since both solve the same equation with the same initial condition $\mathbf{m}(0)=\mathbf{M}(0)=\mathbf{m}_0$. Having constructed the master equation so that $K_i^{(1)\infty}=F_i$ for the F_i's of Eq. (1), we achieve an embedding of the macrovariable equations in the master equation description as the macroscopic limit.

We can also obtain a dynamical description of the intrinsic fluctuations with this master-equation approach. Generally, the intrinsic fluctuations in the macrovariables scale[11] like $1/\Omega^{1/2}$. This means that they simply vanish in the macroscopic limit. In the spirit of the central limit theorem of probability theory,[23] it is possible to rescale the fluctuations so that their limiting behavior may be rigorously deduced. This is done by considering the deviations of the $\mathbf{m}$ components from the deterministic solution to the macroscopic limit equation (12), i.e., $\mathbf{m}(t)$ scaled with $\Omega^{-1/2}$:

$$\mathbf{m}=\mathbf{m}(t)+\Omega^{-1/2}\mu \ , \tag{14}$$

which defines the scaled intrinsic fluctuations μ. This scaling implies that as $\Omega\rightarrow\infty$, the μ components are of order unity. We shift attention from the probability distribution $P(\mathbf{m},t)$ to the probability distribution for the scaled intrinsic fluctuations $\Phi(\mu,t)$. It is then possible to show[2] that in the macroscopic limit (i.e., $\Omega\rightarrow\infty$), we obtain

$$\frac{\partial}{\partial t}\Phi=-\frac{\partial}{\partial\mu_i}\left[\frac{\partial}{\partial m_j}K_i^{(1)\infty}(\mathbf{m}(t))\mu_j\Phi\right] +\frac{1}{2}\frac{\partial^2}{\partial\mu_i\partial\mu_j}[R_{ij}^{(2)}(\mathbf{m}(t))\Phi] \ , \tag{15}$$

in which $R_{ij}^{(2)}$ is defined by

$$R_{ij}^{(2)}=\lim_{\Omega\rightarrow\infty}\Omega K_{ij}^{(2)}(\mathbf{m}(t)) \ . \tag{16}$$

This is a Fokker-Planck equation for a nonstationary, Gaussian, Markov process. The nonstationary results from the explicit dependence on $\mathbf{m}(t)$ in both $K_i^{(1)\infty}$ and $R^{(2)}$. Since this $\mathbf{m}(t)$ is found from (12), the deterministic macrovariable equations, we say that the intrinsic fluctuations "ride on the back" of the deterministic motion. We will refer to the rigorous proof of this result as "Kurtz's *first* theorem."

Several remarks are in order.[2] The time-dependent coefficients of the first-order μ derivatives in Eq. (15) are precisely the components of the Jacobi matrix for the deterministic macrovariable equations [either (1) or (12)]

$$J_{ij}(t)=\frac{\partial}{\partial m_j}K_i^{(1)\infty} \ . \tag{17}$$

Defining the covariance matrix for the intrinsic fluctuations by

$$C_{ik}(t)=\langle\mu_i(t)\mu_k(t)\rangle \ , \tag{18}$$

where $\langle\ \rangle$ denotes averaging with respect to $\Phi(\mu,t)$, leads to the equation [derived from Eq. (15)]

$$\frac{d}{dt}C_{ik}(t)=J_{ij}(t)C_{jk}(t)+C_{ij}(t)J_{kj}(t)+R_{ik}^{(2)}(t) \ . \tag{19}$$

This is exactly (4) of the Introduction [(17) is precisely (2) because of Eq. (6)] and shows how the Jacobi matrix for the deterministic motion arises in the dynamics of the intrinsic fluctuations. The following and final remark is the central issue of this paper. If the deterministic motion is chaotic, then the Jacobi matrix will create an unbounded growth of the C_{ik} components.[2,4] Since the derivation of (15), and hence of (19), assumes that the μ components remain of order unity, it would be inconsistent to use Eq. (19) when the fluctuations grow larger than this. As will be shown below, there exists an alternative treatment[5,6] for this case in which the intrinsic fluctuations can grow large.

One way to express the content of the limit theorem[23] reviewed above is to write

$$\mathbf{M}(t)=\langle\mathbf{m}\rangle_t+O(1/\Omega^{1/2}) \ , \tag{20}$$

in which $\langle \ \rangle_t$ is the average with respect to $P(\mathbf{m},t)$. This says that the deterministic equations' solution approximates the expected values of the underlying mesoscopic master equation with an error of order $1/\Omega^{1/2}$, i.e., an error the size of the fluctuations. The proper interpretation of this result is that the more fundamental physical description is given by the master equation, whereas the deterministic macrovariable equation is an approximate description. In the macroscopic limit where intrinsic fluctuations may be ignored (provided that they do not grow large), it is far easier to use the macrovariable equations than to use the master equation. However, if the intrinsic fluctuations grow too large for this treatment to be valid (seen as chaos at the macrovariable level), then another limit theorem is available, "Kurtz's *second* theorem."[5,6] Not only does Kurtz's second theorem allow one to handle the large intrinsic fluctuations, but it does so with even greater accuracy than expressed in (20). If we denote the solution to this alternative treatment, to be elucidated below, by $\mathbf{M}_f(t)$, then Kurtz's second theorem[5,6] implies

$$\mathbf{M}_f(t)=\langle \mathbf{m} \rangle_t + O(\ln\Omega/\Omega) \ . \tag{21}$$

$\mathbf{M}_f(t)$ combines both the macrovariable behavior and the large fluctuations and its probability distribution satisfies the Fokker-Planck equation

$$\frac{\partial}{\partial t}P_f(\mathbf{m},t)=-\frac{\partial}{\partial m_i}[K_i^{(1)\infty}(\mathbf{m})P_f(\mathbf{m},t)] + \frac{1}{2}\frac{\partial^2}{\partial m_i \partial m_j}[K_{ij}^{(2)\infty}(\mathbf{m})P_f(\mathbf{m},t)] \ , \tag{22}$$

so that

$$\mathbf{M}_f(t)=\int d^N m \ \mathbf{m} P_f(\mathbf{m},t) \ . \tag{23}$$

When this limit theorem was originally obtained,[5,6] the chaotic amplification of intrinsic fluctuations was not yet clearly understood.[9,10] Since the typical applications involved near equilibrium states or stable steady states away from critical points, for which intrinsic fluctuations remained small, a vanishingly small difference in behavior resulted from using (22) instead of the more tractable (15). Thus this treatment remained largely ignored. On occasion, however, an objection to (22) has been voiced[12] because the averaging defined by (23) implies

$$\frac{d}{dt}(M_f(t))_i=\langle K_i^{(1)\infty}(\mathbf{m})\rangle \neq K_i^{(1)\infty}(\langle \mathbf{m} \rangle) \tag{24}$$

since Eq. (22) does not have a Dirac δ-function solution [cf. Eqs. (10)–(13)]. For intrinsic fluctuations that remain small, the difference between the two expressions on the right-hand side of (24) is only order $O(1/\Omega^{1/2})$, i.e., ignorable. For intrinsic fluctuations that grow large, this same inequality is a sign of the breakdown of the macrovariable limit altogether, as has been shown earlier.[1,2] Therefore, Eq. (22) is perfectly suited to the situation we are confronting.

Because the direct solution to (22) is numerically demanding, we prefer to use a more tractable, equivalent[9] method, the nonlinear Langevin treatment. This is possible because to every probability distribution equation satisfying (22), there is associated a unique Langevin-like equation. However, great care is required in order to express the Langevin equivalent correctly, since there are two valid, yet distinct versions of stochastic calculus by which the equivalence can be realized, the Ito and the Stratonovich versions.[26] The proof of the limit theorem[5,6] that produces Eq. (22) makes use of Martingale properties[26] and in so doing arrives at Eq. (22) in the Ito context. Numerical realizations of Langevin equations in our hands are done in the manner of Stratonovich[27] using the traditional Newtonian calculus. Therefore, we need to obtain the Stratonovich Langevin equation equivalent to the Ito probability distribution Eq. (22). This is done as follows. Suppose $\mathbf{M}_f(t)$ satisfies the stochastic differential equation

$$\frac{d}{dt}(\mathbf{M}_f(t))_i=\alpha_i(\mathbf{M}_f(t))+\beta_{ij}(t)g_j(t) \ , \tag{25}$$

where the derivatives are to be manipulated according to the usual calculus and where the g_j's are statistically independent Gaussian white noises with zero means and covariances of unit strength, i.e.,

$$\langle g_k(t)\rangle = 0 \ , \tag{26}$$

$$\langle g_i(t)g_k(t')\rangle=\delta_{ik}\delta(t-t') \ , \tag{27}$$

in which $\langle \ \rangle$ denotes averaging with respect to the g_k distributions. The Fokker-Planck equation satisfied by the Stratonovich stochastic process in Eq. (25) is[26]

$$\frac{\partial}{\partial t}P_f(\mathbf{m},t)=-\frac{\partial}{\partial m_i}[\alpha_i(\mathbf{m})P_f(\mathbf{m},t)] + \frac{1}{2}\frac{\partial}{\partial m_i}\beta_{ik}(t)\frac{\partial}{\partial m_j}\beta_{jk}(t)P_f(\mathbf{m},t) \ , \tag{28}$$

which may be rearranged as

$$\frac{\partial}{\partial t}P_f(\mathbf{m},t)=-\frac{\partial}{\partial m_i}\left[\alpha_i(\mathbf{m})P_f(\mathbf{m},t) + \frac{1}{2}\left[\frac{\partial}{\partial m_j}\beta_{ik}(t)\right]\beta_{jk}(t)P_f(\mathbf{m},t)\right] + \frac{1}{2}\frac{\partial^2}{\partial m_i \partial m_j}\beta_{ik}(t)\beta_{jk}(t)P_f(\mathbf{m},t) \ . \tag{29}$$

In both (28) and (29), repeated indices are summed. To identify the correct α and β to be used in Eq. (25), we need only compare Eqs. (22) and (29). Since $K_{ij}^{(2)\infty}$ is a symmetric, non-negative matrix at each instant of time, the square root of $K_{ij}^{(2)\infty}$ will also be symmetric and one finds that

$$\beta(t)=[K^{(2)\infty}(t)]^{1/2} \ , \tag{30}$$

$$\alpha_i(t)=K_i^{(1)\infty}(t)-\frac{1}{2}\left[\frac{\partial}{\partial m_j}\beta_{ik}(t)\right]\beta_{jk}(t) \ . \tag{31}$$

Generally, β is of order $1/\Omega^{1/2}$ so that α differs from $K_i^{(1)\infty}$ only to order $1/\Omega$ and this "Ito-Stratonovich shift" is ignorable,[9] but when the intrinsic fluctuations are large, not only will this difference be important, but (25) will differ markedly from the purely deterministic macrovariable equation (12) [equivalently (1)].

There is an additional advantage to using Eq. (25) for the study of chaotically amplified intrinsic fluctuations. The only feature of the underlying mesoscopic master equation that remains in Eqs. (25), (30), and (31) is the matrix $K_{ij}^{(2)\infty}$ (the vector $K_i^{(1)\infty}$ is predetermined by the macrovariable equations). Thus we need not know the underlying master equation in full detail, but only the second moment of the transition probability [see (8)]. With physical insight, it may be possible to correctly guess $K_{ij}^{(2)\infty}$ without obtaining the full master equation. Hydrodynamics may be an example of this circumstance.[9]

The description of large-scale intrinsic fluctuations by Eqs. (25)–(27), (30), and (31) combines the macrovariable and the intrinsic fluctuation dynamics in one quantity $\mathbf{M}_f(t)$, unlike the situation for small fluctuations wherein two sets of equations [Eqs. (12) and (15)] are obtained. The intrinsic fluctuations no longer "ride on the back" of the deterministic macrovariables and, indeed, no autonomous macrovariable equation exists [see (24)]. When the intrinsic fluctuations grow large, the distribution function $P_f(\mathbf{m},t)$ becomes broadly spread out, unlike the extremely sharp distribution given by (11), which is only valid when the fluctuations remain small.[9] For this reason, the concept of a deterministic macrovariable is lost. While one may still use (23) to define an "average" value, there is no longer an autonomous dynamics for the $\mathbf{M}_f$ components because of the broadness of the P_f distribution.[2]

The breakdown of the autonomous macrovariable equations associated with large-scale intrinsic fluctuations forces a reassessment of the meaning of chaos in real physical systems. Conceptually, one must shift focus from the wild deterministic macrovariable trajectories to large-scale intrinsic fluctuations. A variety of new characterizations needs to be developed, and the examples that are presented in Sec. IV are meant to indicate some possible avenues for this development. In each of the examples, we will use the approach represented by Eq. (25), since it is the most tractable and is also a highly accurate representation of the mesoscopic level of description.

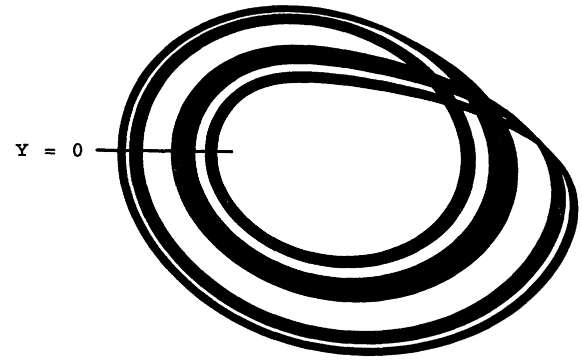

FIG. 1. X-Y plot of the Rossler attractor for $\mu=4.23$ and $\sigma=0$.

One should not confuse this approach with previous work that treats the effects of *extrinsic* noise on macrovariable systems[13,17] using similar equations. In these treatments, some of which have the same form as (25), the α_i's are just the $K_i^{(1)\infty}$'s [i.e., the F_i's of Eq. (1)] and the β_{ik}'s are not connected to the state of the system, i.e., there is no "intrinsic fluctuation-dissipation relation" as in (30), because the fluctuations are *extrinsic* and not *intrinsic.* That is, the strength of the extrinsic noise does not depend on the state of the system. Moreover, if the intrinsic fluctuations have grown by a large scale, the breakdown of the autonomous macrovariable equations implies that extrinsic fluctuations should be introduced directly at the mesoscopic level, not at the deterministic macrovariable level, which is no longer valid.

All the preceding considerations must be qualified by the observation that the growth of intrinsic fluctuations depends upon two quantities, their rate of growth (this is related to the largest Liapunov exponent) and their initial size [this is determined by (30) at $t=0$]. In the Josephson-junction[15] example that follows, both of these quantities are "large," whereas in the laser[16] example,

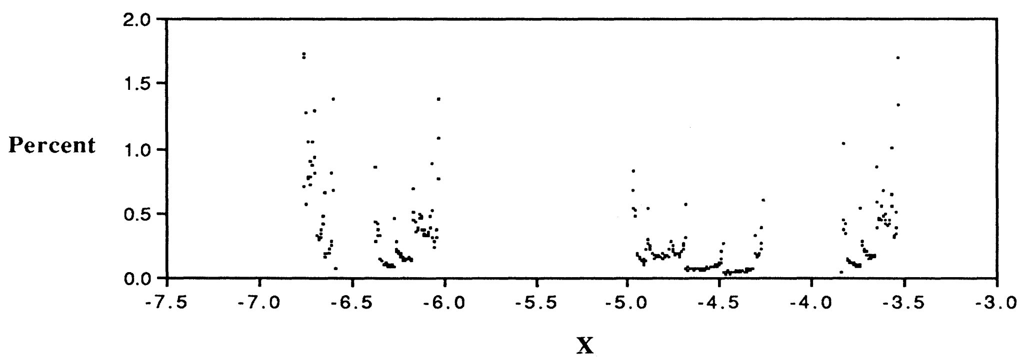

FIG. 2. Invariant measure for the attractor in Fig. 1 projected onto the negative X axis at $Y=0$. The vertical axis gives the percentage of crossing points in an X-axis bin. 1024 bins were used over the range of X values indicated in the figure.

both of these quantities are "small." In the Rossler[14] paradigm, we explore both regimes and motivate our expectations for the real physical systems.

IV. EXAMPLES

The purpose of these examples is twofold. They make the general ideas concrete and they help to make contact with real experiments. Ultimately, we wish to identify a real physical system in which quantitative measurements can be used to explore the amplification of intrinsic fluctuations. Significant progress in this direction is reported.

As our first example, which exhibits behavior like both of the following examples, we look at a purely mathematical model, the Rossler model.[14,28] This model was invented to show the minimal ordinary differential equation system that can have chaos. We have chosen it because of its great simplicity. The route to chaos in this model is period doubling of a limit cycle. The equations, in three independent variables, X, Y, and Z, are

$$\frac{d}{dt}X=-(Y+Z)\ , \tag{32}$$

$$\frac{d}{dt}Y=X+\tfrac{1}{5}Y\ , \tag{33}$$

$$\frac{d}{dt}Z=\tfrac{1}{5}+Z(X-\mu)\ , \tag{34}$$

in which μ is an adjustable parameter. For $\mu=2.6$, the asymptotic state is a simple limit cycle attractor. It has a period of about 5.8 time units. The unit of time is dimensionless, and power spectra show a fundamental at about 0.17 Hz (cycles per unit of dimensionless time). (In the literature,[14] the unit of time is arbitrarily taken to be 0.01 s, so that the fundamental becomes 17 Hz.) For $\mu=3.5$, the limit cycle has bifurcated once, while for $\mu=4.1$, it has done so twice. After this, much smaller changes in μ lead to increasing numbers of bifurcations; until around $\mu=4.2$, infinitely many have occurred and the motion becomes chaotic. For $\mu=4.23$, the largest Liapunov exponent is $\lambda=0.014$.

This system of equations does not describe a real physical system. Therefore, construction of an underlying master equation cannot benefit from physical insight into real molecular substructure. Nevertheless, for the sake of illustration, we can imagine that such an underlying, mesoscopic, molecular picture really does exist. This means that we must construct an underlying master equation for the Rossler model, based on an imagined underlying molecular basis. There are many ways to do this that yield the Rossler model in the macroscopic limit but produce different fluctuations. Whichever specific choice we make, we can circumvent the actual construction of the master equation by invoking Kurtz's second theorem. We do so by merely adding an *intrinsic* noise term to Eq. (34), say, in accord with Kurtz's second theorem as discussed in Sec. III. While arbitrary for the Rossler model, this procedure serves to illustrate how noise amplification can be seen in models of real physical systems, wherein the specification of the added noise is determined entirely by the nature of the physical system. The noise to be added to the Rossler model is Gaussian, white noise with state independent strength, so that no Ito-Stratonovich shift is required.

Note that what we are doing looks similar to what others have done to treat the addition of *extrinsic* noise to the Rossler model. However, the interpretation is significantly different. For extrinsic noise, X, Y, and Z re-

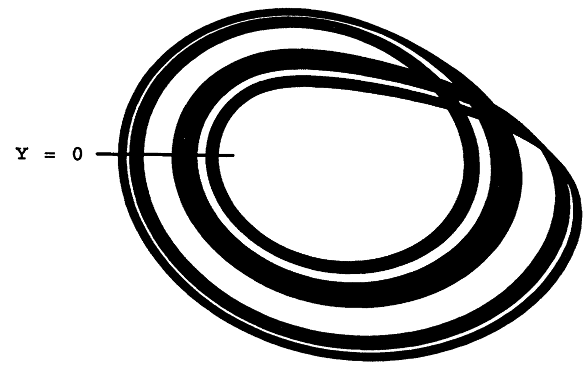

FIG. 3. X-Y plot of the Rossler attractor for $\mu=4.23$ and $\sigma=10^{-8}$.

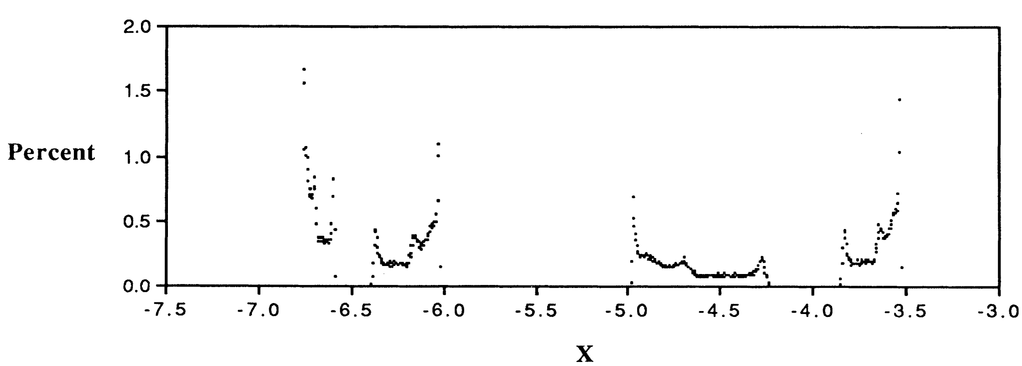

FIG. 4. Probability distribution for the attractor in Fig. 3 projected onto the negative X axis at $Y=0$. All other aspects of the figure are the same as in Fig. 2.

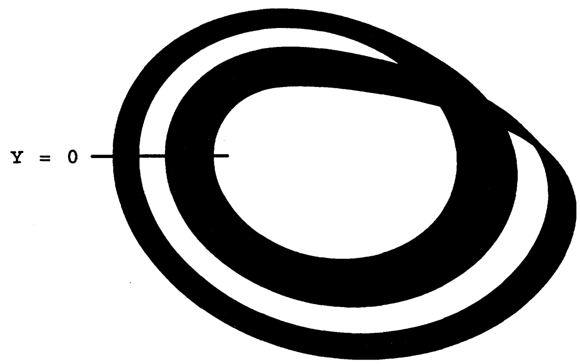

FIG. 5. X-Y plot of the Rossler attractor for $\mu=4.23$ and $\sigma=10^{-6}$.

tain their meaning and their values merely become noisy, but for intrinsic molecular noise, the underlying probability distribution implicitly in mind when we construct the mesoscopic description, either by a master equation or by Kurtz's second theorem, becomes broad because of chaos amplification of noise, and X, Y, and Z cease to be meaningful variables. No autonomous dynamics exists for them. In other words, the macrovariable picture breaks down,[2,4] and the mesoscopic description is required for a correct quantitative treatment.

Let us now return to our *ad hoc* mesoscopic treatment of the Rossler model. The observation of the amplification of intrinsic noise by chaotic trajectories is achieved in the following manner: First, we run Eqs. (32)–(34) numerically and plot the attractor (after the transients have died away) in the X-Y plane (X along the horizontal axis and Y along the vertical axis). This is shown in Fig. 1 for $\mu=4.23$. Also shown is a horizontal line cutting the left-hand portion ($X<0$) of the attractor along $Y=0$. We determine numerically the probability distribution (in the noise-free case, this is called the invariant measure) for X values. This is shown in Fig. 2. Next, we redo all of this with the noise present. As indicated above, this is done by using Eqs. (32) and (33) as is, and by adding Gaussian, white noise with zero mean g to Eq. (34), i.e.,

$$\frac{d}{dt}Z=\tfrac{1}{5}+Z(X-\mu)+g\ , \tag{35}$$

in which g has correlation formula

$$\langle g(t)g(t')\rangle=2\sigma\delta(t-t')\ , \tag{36}$$

in which σ is an adjustable noise strength. In a real physical model, this noise strength would be determined by the underlying physics through the master equation. For our illustrative purposes, it is adjustable so that we can explore how effects depend on its size. Figures 3 and 4 show the results paralleling Figs. 1 and 2 for $\mu=4.23$ and $\sigma=10^{-8}$. It is extremely difficult to discern any differences between Figs. 1 and 3, but there is very clear smoothing of the probability distribution of Fig. 2 in Fig. 4 as a result of intrinsic noise amplification. If, instead, our noise has been instrumental, then we would see it as a smoothing of Fig. 2 with a Gaussian smoothing function with standard deviation equal to $\sigma^{1/2}$, a magnitude of 10^{-4}, that would not produce a visually observable effect on Fig. 2. However, amplification of intrinsic noise produces the clearly observable effect seen in Fig. 4 and shows that the amplification is to macroscopic size (i.e., order unity). Figures 5 and 6 show what happens when $\sigma=10^{-6}$. Now both figures are visually effected and the attractor shows only two bands instead of four. The attractor in Fig. 5 could be mistaken for the more chaotic, noise-free attractor in Fig. 7 obtained for $\mu=4.3$, but the corresponding invariant measure of Fig. 8 is easily distinguished from Fig. 6.

These cases clearly suggest that the way to observe the chaotic amplification of intrinsic noise is to contrast the resulting probability distribution with the noise-free invariant measure. Even when the corresponding attractor plots show no discernible differences, the differences in the probability distributions can be very marked. For big enough noise, even the attractor plots may become distinguishable. The following two examples illustrate this diagnostic approach in models of real physical systems.

The Josephson junction is a real, electronic, physical system in which conditions can be arranged so that it appears to exhibit chaos. A simple mathematical description of the phenomenon in terms of either a macrovariable current, or a macrovariable voltage (or associated

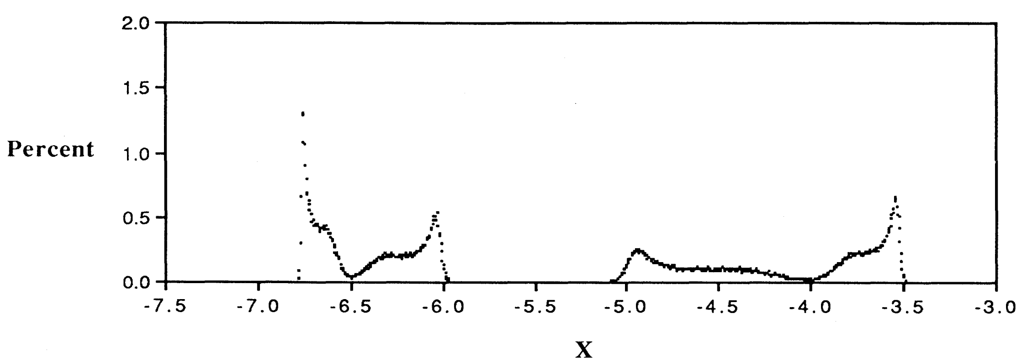

FIG. 6. Probability distribution for the attractor in Fig. 5 projected onto the negative X axis at $Y=0$. All other aspects of the figure are the same as in Figs. 2 and 4.

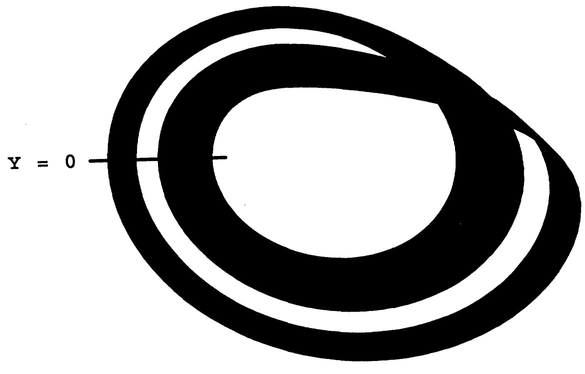

FIG. 7. X-Y plot of the Rossler attractor for $\mu=4.3$ and $\sigma=0$.

phase), also can exhibit chaos. Incidentally, this is one of those examples, alluded to in the Introduction, for which published accounts[15] refer to the chaos in the macrovariable time dependence as a "noise rise." This usage is not what we mean by "chaotically amplified intrinsic noise," and one must make an effort to avoid confusion.

The macrovariable model for superconductor-insulator-superconductor (SIS) Josephson junctions operated in the classical regime (i.e., $eI_0R < k_BT$ to be interpreted below) is[15]

$$C\frac{dV}{dt}+\frac{V}{R}+I_0\sin\phi=I_{\mathrm{dc}}+I_{\mathrm{rf}}\sin\omega t \ , \tag{37}$$

in which ϕ is the macroscopic quantum phase of the supercurrent, C is the capacitance of the junction, R is its resistance, I_0 is the critical current, I_{dc} is the applied dc current, I_{rf} is the amplitude of the applied rf current with frequency ω, and V is the junction voltage related to ϕ by

$$V=\frac{\hbar}{2e}\frac{d\phi}{dt} \ , \tag{38}$$

in which $\hbar$ is Planck's constant (divided by 2π) and e is the charge of an electron. One may proceed with the two coupled equations (37) and (38), or convert to one second-order equation

$$\frac{\hbar}{2e}\frac{C}{I_0}\frac{d^2\phi}{dt^2}+\frac{\hbar}{2e}\frac{1}{I_0R}\frac{d\phi}{dt}+\sin\phi=\frac{I_{\mathrm{dc}}}{I_0}+\frac{I_{\mathrm{rf}}}{I_0}\sin\omega t \ . \tag{39}$$

This form of the equation suggests defining the junction frequency ω_0 by

$$\omega_0=\left[\frac{\hbar C}{2eI_0}\right]^{-1/2} \tag{40}$$

and the dimensionless time τ by

$$\tau=\omega_0 t \ . \tag{41}$$

If we also introduce the McCumber parameter[29] $\beta_c=2eI_0R^2C/\hbar$ and the ratios $\rho=I_{\mathrm{dc}}/I_0$ and $\rho_1=I_{\mathrm{rf}}/I_0$, Eq. (39) becomes

$$\frac{d^2\phi}{d\tau^2}+\frac{1}{\sqrt{\beta_c}}\frac{d\phi}{d\tau}+\sin\phi=\rho+\rho_1\sin\left[\frac{\omega}{\omega_0}\tau\right] , \tag{42}$$

which is the canonical form for the Josephson junction and is seen to be the equation for a periodically perturbed, damped, planar pendulum,[30] well known for its capacity to exhibit chaos.

This description of the junction is macroscopic and the macrovariable current represents many Cooper electron pairs. Individual Cooper-pair motions show up as intrinsic fluctuations in the macrovariable current. This is not unlike the picture of current fluctuations in a classical resistor,[9] i.e., Johnson noise, except that the electrons are not paired and, in addition, Johnson noise occurs in a resistor in series with a voltage, whereas Josephson-junction noise occurs with a resistor and a capacitor in parallel with the junction voltage.

In principle, we should now try to construct a master equation that has Eq. (42) as its macroscopic limit and contains the correct physics for the determination of $K^{(2)\infty}$. This is not an easy task. However, in other electronic circuits with a configuration of capacitor and resistance identical to that for Eq. (37) (i.e., in parallel with the voltage), the determination of the strength of the fluctuations through a master equation, has already been obtained successfully.[9] This allows us to use Kurtz's second theorem to obtain a stochastic realization of the

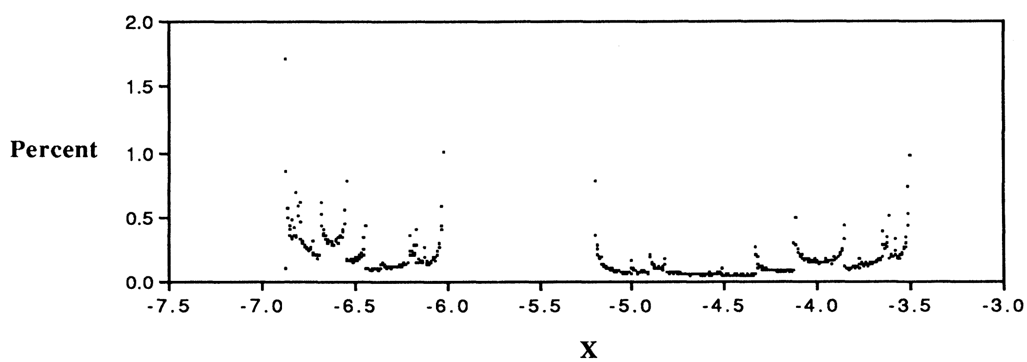

FIG. 8. Invariant measure for the attractor in Fig. 7 projected onto the negative X axis at $Y=0$. All other aspects of the figure are the same as in Figs. 2, 4, and 6.

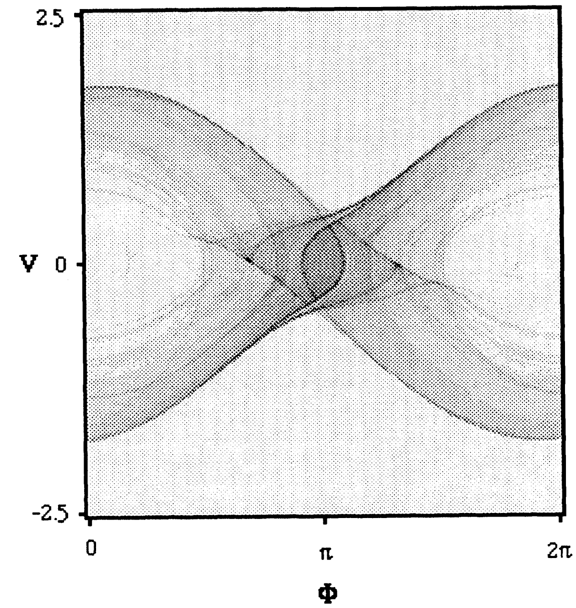

FIG. 9. Invariant measure for the Josephson-junction equations with no noise.

mesoscopic description. The result is to add a stochastic term to the right-hand side of (37) of the form $g\,f(t)$, where f is Gaussian, white noise with zero mean and

$$\langle f(t)f(t')\rangle=\delta(t-t')\ , \tag{43}$$

$$g=(2k_BT/R)^{1/2}\ , \tag{44}$$

in which k_B is Boltzmann's constant, T is the junction temperature, and R is the junction resistance. (Note that for Johnson noise, $g\sim R^{1/2}$ when quantities are expressed as functions of frequency instead of time.) This amounts to the addition of $(g/I_0)\omega_0^{1/2}f(\tau)$ to the right-hand side of (42), where

$$\langle f(\tau)f(\tau')\rangle=\delta(\tau-\tau')\ . \tag{45}$$

Since the numerical integration of this nonintegrable equation is easier to implement as two coupled first-order equations, we recast it as

$$\frac{d\phi}{d\tau}=v\ , \tag{46}$$

$$\frac{dv}{dt}+\frac{1}{\sqrt{\beta_c}}v+\sin\phi=\rho+\rho_1\sin\left[\frac{\omega}{\omega_0}\tau\right]+\left[2\frac{I_T}{I_0}\right]^{1/2}\beta_c^{-1/4}f(\tau)\ , \tag{47}$$

where (46) defines the variable v, and in Eq. (47) we have introduced the "thermal current" I_T defined by

$$I_T=\frac{2ek_BT}{\hbar} \tag{48}$$

and have used the identity

$$\frac{g\omega_0^{1/2}}{I_0}=\left[2\frac{I_T}{I_0}\right]^{1/2}\beta_c^{-1/4}\ . \tag{49}$$

We see from (49) that the fluctuation-dissipation relation maintains its usual significance in this case because the mean square of the fluctuation has a strength proportional to both $2k_BT$ and $\beta_c^{-1/2}$. Moreover, it is inversely proportional to the system size, in this case I_0, which itself is proportional to the cross-sectional area of the junction. The cross-sectional area of the junction is the macroscopic parameter, i.e., Ω, characteristic of this system. A particularly nice feature of this example is that the fluctuation strength is independent of the state of the system (insofar as R is). This is why there is no "β correction to α" [see Eqs. (30) and (31)] in (47). Said another way, the Ito-Stratonovich distinction is irrelevant in this case.

We have done numerical studies of Eqs. (42), (46), and (47). The results are planned to be reported in detail elsewhere.[31] Using physically derived parameters ($\beta_c=4$, $\rho=0$, $\rho_1=0.91$, and $\omega/\omega_0=0.5655$), the scaled parameters in (47) are all roughly of order unity, except for the noise strength given in (49). It works out to be of order 10^{-2}. There is no freedom here because this magnitude is determined by the fluctuation-dissipation relation expressed by (44) and depends on predetermined macroscopic parameters (i.e., T and R). This magnitude is relatively very large. For comparison, a typical hydrodynamics problem cast in dimensionless form, such that the macrovariable magnitudes are order unity, has a

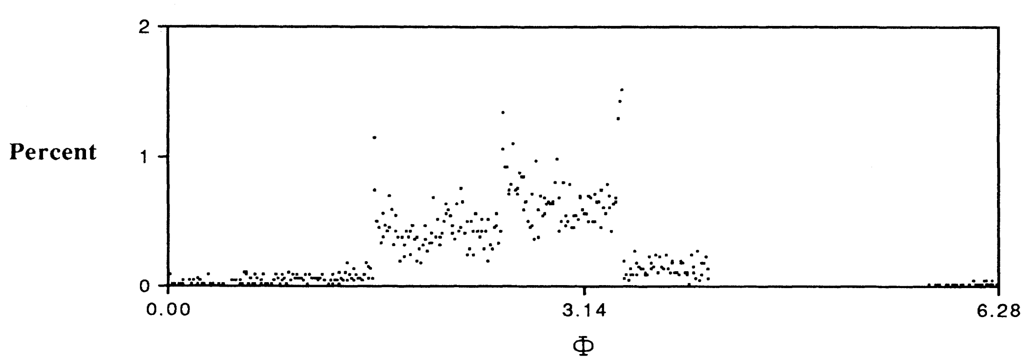

FIG. 10. Projection of Fig. 9 along with $v=0$ axis yielding a the ϕ distribution.

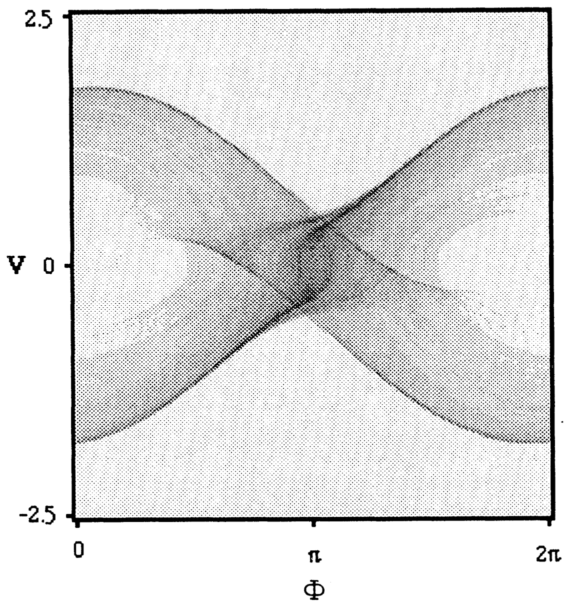

FIG. 11. Probability distribution for the Josephson junction with intrinsic noise.

mean-squared noise strength of order 10^{-10}. Moreover, the largest Liapunov exponent for (42) with the same parameters is $\lambda=0.112$, which implies a sizable amplification of the intrinsic noise in only 10–100 dimensionless time units. This does show up in the attractor plots with the noise compared with those with no noise (see Figs. 9 and 11). This is like the Rossler case of $\mu=4.23$ with $\sigma=10^{-6}$. In addition, dramatic differences in the probability distributions are seen, as is shown in Figs. 10 and 12.

Recent studies of a class-*B* Nd:YAG laser containing a nonlinear intracavity crystal exhibited chaotic output intensity.[16] The dynamics was shown to be very well modeled by equations such as

$$\tau_c \frac{dI_j}{dt} = \left[G_j - \alpha_j - g\epsilon I_j - 2g\epsilon \sum_{k \neq j}^{3} I_k \right] I_j \ , \qquad (50)$$

$$\tau_f \frac{dG_j}{dt} = G_j^0 - G_j \left[1 + \beta_j I_j + \sum_{k \neq j}^{3} \beta_{jk} I_k \right] \ , \qquad (51)$$

for $j,k=1,2,3$. These equations represent only one of many possible cases studied. In this case, three modes polarized in the same direction have intensities I_j and gains G_j for $j=1,2,3$. In other cases, six, or even eight modes are used and the equations are correspondingly enlarged. The cavity round-trip time τ_c is set equal to 0.2 ns, the fluorescence time τ_f is set equal to 240 μ s, the cavity losses α_j are set equal to 0.01, the nonlinear crystal coupling coefficient ϵ is set equal to 5×10^{-5}, the self-saturations β_k are each set equal to 1, the cross saturations β_{jk} are each set equal to 0.6 and the pump parameters G_j^0 are each set equal to 0.04. The parameter g is a variable configuration parameter depending on the relative orientation of the laser and nonlinear crystals. For different choices (g is always in the interval [0,1]), stable, periodic, chaotic, and intermittent output intensities are produced. The correspondence between the numerical simulation of Eqs. (50) and (51) and real laser measurements for which all of the above parameters were determined is good in the periodic regime when the time course of the total intensity is compared. Spontaneous emission is the physical basis for intrinsic noise in this laser system (pump noise may also prove important, but appears to be very small in this case), and in other laser contexts,[32] it has been very accurately simulated by adding Gaussian, white noise to equations that are the analogs to Eqs. (50) and (51). We may do the same here, in the spirit of Kurtz's second theorem.

Chaos is confirmed for the equations by computing the Liapunov exponent, which turns out to be $\lambda=4.6\times10^4$ s^{-1}.[33] The magnitude of the white noise that should be used to model spontaneous emission is of order 10^{-8}. The probability distribution for the total intensity shows a significant effect in our preliminary studies, and this characterization is currently under investigation. A detailed account of the comparison of the theory with experiment is in preparation.[33]

Generally, a numerical simulation of model equations will determine whether or not amplification of intrinsic noise will be significant. If the initial intrinsic noise level is n_0 and the largest Liapunov exponent is λ, then the time required for the noise level to reach n is of the order of[4]

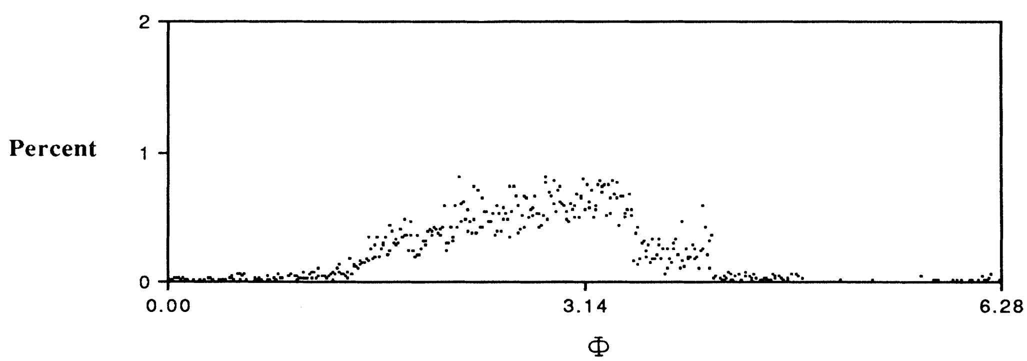

FIG. 12. Projection of Fig. 11 along the $v=0$ axis yielding *a* the ϕ distribution.

$$t = \frac{1}{\lambda} \ln \left[\frac{n}{n_0} \right] . \tag{52}$$

It may take much longer because this value assumes pure exponential growth, whereas after a certain noise level, nonlinearities will begin to suppress the noise growth.

ACKNOWLEDGMENTS

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