Stochastic effects in Rayleigh-Bénard pattern formation

T. C. Elston and Ronald F. Fox
School of Physics, Georgia Institute of Technology, Atlanta, Georgia
(Received 15 July 1991)

Results of recent experiments on the Rayleigh-Bénard system indicate that stochastic driving forces are important in the evolution of flow patterns. However, the strength of the noise needed to reproduce the experimental results is four orders of magnitude larger than that of thermal noise. In this report, we present evidence that suggests that the source of noise comes from an uncertainty in the initial conditions rather than from a stochastic driving force.

PACS number(s): 47.20.Ky, 05.40.+j

The Rayleigh-Bénard problem of heating a fluid from below is a classic example from hydrodynamics in which a nonlinear dissipative system undergoes a transition from a spatially uniform state to one of lower symmetry. The important parameter in this problem is the Rayleigh number R, which is proportional to the temperature gradient ΔT across the fluid. Below its critical value R_c, a pure conduction state exists in the fluid with no velocity field present. When R is greater than R_c, the conduction state is unstable and any small perturbation will cause the onset of convection. In most experimental systems, convection will actually occur when R is below R_c because of imperfections in the experimental apparatus. The main source of these imperfections comes from thermal gradients at the boundary due to the differences in thermal diffusivities of the container wall and the fluid. This sidewall forcing not only causes subcritical bifurcations, but also causes the emerging flow patterns to possess the same symmetry as the container. In an attempt to study the effect of stochastic processes on pattern evolution, Meyer and co-workers [1] have constructed an experimental cell whose walls are made from a gel with a thermal diffusivity which is almost identical to that of water, thus eliminating sidewall forcing. When this was done the flow patterns that emerged had randomly oriented convection cells that were not reproduced on subsequent runs of the experiment. These results imply that stochastic processes play an important role in pattern formation.

A more convincing argument for the importance of stochastic processes in this system is found through measurements of the convective heat current j^{conv}. The bifurcation from the conductive to the convective state and the amplitude of the resulting velocity field for R slightly greater than R_c are described by the Landau amplitude equation,

$$\tau_0 \frac{dA}{dt} = \epsilon A - A^3 + h \ , \tag{1}$$

where ϵ is given by $R/R_c - 1$ or $\Delta T/\Delta T_c - 1$, and h represents any imperfections in the system. τ_0 is used to scale time to the vertical thermal diffusion time and for the case under consideration, $\tau_0 = 0.0552$. The $A = 0$ conductive state is a stable solution to the Landau equation when $\epsilon < 0$. When ϵ becomes positive, this solution is unstable and the resulting convective heat current can be found from $j^{\text{conv}} = A^2$. In the experiments performed by Meyer, Ahlers, and Cannell [2], ϵ was ramped linearly in time, i.e., $\epsilon = \epsilon_0 + \beta t$. Using the Landau amplitude equation, Meyer, Ahlers, and Cannell were able to very accurately reproduce experimental data obtained for the convective heat current with h the only adjustable quantity. They found a reasonable fit to the data was obtained with h taken to be constant, however, a stochastic h (stochastic driving force) produced noticeably better results. While this provided convincing evidence for the presence of a stochastic process, the strength of the stochastic force needed to fit the data was four orders of magnitude larger than the strength of the thermal noise predicted by fluctuating hydrodynamics [3,4]. The source of this noise has remained a mystery.

At the suggestion of Rabinovich [5], we decided to investigate the possibility that the noise source in this experiment was an uncertainty in initial conditions rather than a stochastic driving force. Two ramp rates for ϵ were studied, $\beta = 0.27$ and 0.08. For each value of β, the Landau amplitude equation with a time-dependent ϵ was numerically integrated for three different cases. In the first run, h was taken to be a deterministic constant force. For $\beta = 0.27$, $h = 1.10 \times 10^{-4}$ and for $\beta = 0.08$, $h = 1.2 \times 10^{-4}$, which are the same values as those used by Meyer, Ahlers, and Cannell [2]. The second run was preformed with h being a Gaussian white noise driving force. In each case $\langle h(t)h(t') \rangle = 2d\tau_0\delta(t-t')$, where $d = 5.3 \times 10^{-7}$ for $\beta = 0.27$ and $d = 6 \times 10^{-7}$ for $\beta = 0.08$, which are again identical with the work by Meyer, Ahlers, and Cannell [2]. A final run was made with random initial conditions and no forcing term. The initial value distribution was Gaussian with average value 0 and variance of 8×10^{-6} for $\beta = 0.27$ and 10^{-5} for $\beta = 0.08$. In Fig. 1, we show the results for the case $\beta = 0.27$. The solid line represents the deterministic case and the dashed line represents both the stochastically driven and random initial condition cases since the two curves are nearly identical. Figure 2 is the same as Fig. 1 except with $\beta = 0.08$. Once again there is virtually no difference between the stochastically driven and the random initial condition cases. In Figs. 3 and 4, we show plots of the standard deviation of j^{conv} as a function of time for

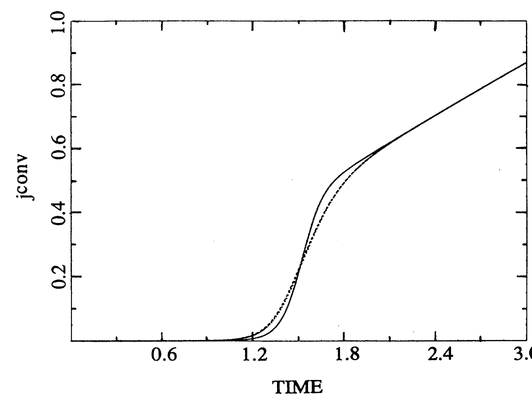

FIG. 1. The convective heat current as a function of time for $\beta=0.27$. The solid line is the deterministic case and the dashed line corresponds to both the stochastic driving force and the random initial condition cases. The time is measured in units of the thermal diffusion time.

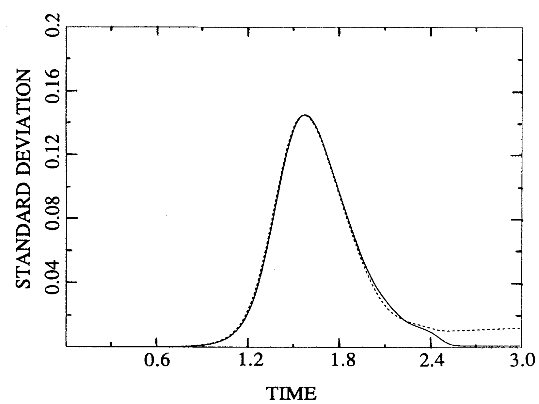

FIG. 3. The standard deviation of the convective heat current as a function of time for $\beta=0.27$. The solid line corresponds to random initial conditions and the dashed line to a stochastic driving force.

$\beta=0.27$ and 0.08 respectively. Here, the solid line represents the random initial condition case, and the stochastically driven case is shown as the dashed line. While slight differences can now be seen, the two curves are very similar. All averages were taken over 10 000 realizations to ensure adequate statistics.

The only notable difference between the work of Meyer, Ahlers, and Cannell [2] and our own is that in their simulations the actual ramp rates varied in time and in ours were constant. This difference, however, only affected the late time magnitudes slightly and none of the qualitative features. We have shown that the source of noise in this experiment need not be attributed to a stochastic driving force, but may well come from an uncertainty in initial conditions. We suspect the initial value fluctuations we get from fitting the data represent some systematic uncertainty in the experimental setup. One possibility is suggested from consideration of the measurement technique used in the experiment. While we do 10 000 stochastic realizations to get a mathematical fit, only a single experiment is performed. This difference is explained [6] by the observation that in this single experiment, a Nusselt-number measurement is performed which effectively integrates the heat transport by many individual patches of convection in different parts of the container. If the detailed states for each patch initially show a variation in the amplitude A comparable with the initial value distribution we have used, then this may be the underlying cause.

It should be noted, however, that the magnitude of the initial value fluctuations is much larger than one would expect from thermal fluctuations. In all of the work

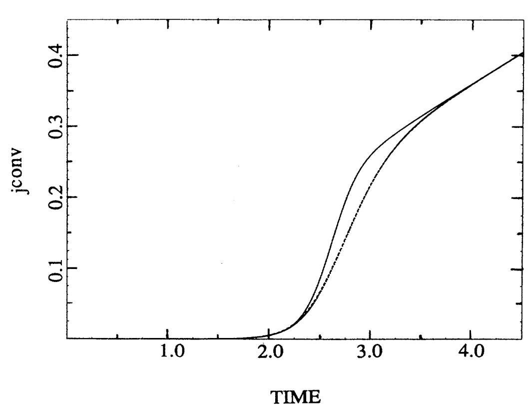

FIG. 2. The convective heat current as a function of time for $\beta=0.08$. The solid line is the deterministic case and the dashed line corresponds to both the stochastic driving force and the random initial condition cases. The time is measured in units of the thermal diffusion time.

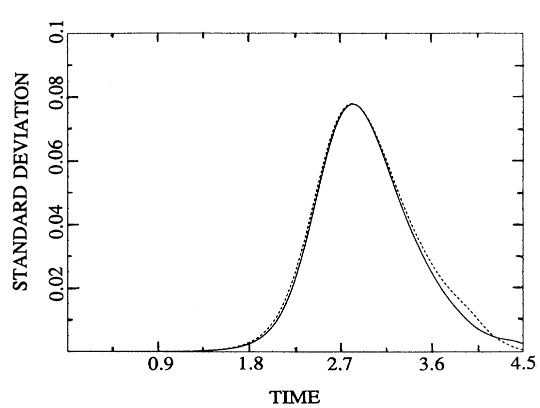

FIG. 4. The standard deviation of the convective heat current as a function of time for $\beta=0.08$. The solid line corresponds to random initial conditions and the dashed line to a stochastic driving force.

quoted above, the origin of time *in the figures,* $t=0$, was chosen as the time when $\epsilon=0$. In fact, Meyer, Ahlers, and Cannell generally took $\epsilon_0=-\beta$ in $\epsilon=\epsilon_0+\beta t$ so that the system aged for a unit time interval before $\epsilon=0$. Thus, *in the equations,* the origin of time is the time when the ramping begins in $\epsilon=\epsilon_0+\beta t$. How does this choice affect the outcome and how does it relate to the size of the initial value thermal fluctuations?

Since $\epsilon_0<0$, the nonlinear term in (1) remains very small and may be ignored during this aging process. The appropriate equation for this stage $(0<t<1)$ is

$$\tau_0\frac{dA}{dt}=(-\beta+\beta t)A+h\ . \tag{2}$$

First consider using only an initial value distribution, i.e., $h=0$. The variance of A grows in accord with (since the mean value remains zero all the time)

$$\langle A^2(t)\rangle=\exp(-2\beta t+\beta t^2)\langle A^2(0)\rangle\ . \tag{3}$$

At time $t=1$ (i.e., when $\epsilon=0$), this yields a factor of $\exp(-\beta)$. As compared with our results for the origin of time used in the figures, taken when $\epsilon=0$, we need only adjust our initial value variance ($\sim 10^{-5}$) by $\exp(\beta)$ in order to get identical results if we start at the earlier time, i.e., at the onset of ramping. This is just a factor of order unity and creates no significant effect.

In order to consider the size of initial-value thermal fluctuations, we must consider another type of aging. The system must be allowed to achieve thermal equilibrium before the ramping begins. During this aging process, the value of ϵ is just the constant ϵ_0 because the ramping has not yet begun. If the only source of initial value fluctuations is thermal fluctuations, then the value for $\langle A^2(0)\rangle$ to be used in (3) would be obtained from the equation

$$\tau_0\frac{dA}{dt}=-\beta A+h_T\ , \tag{4}$$

with h_T representing thermal fluctuations and the origin of time now taken in the very distant past. The variance for h used earlier, d, was of order 10^{-7} to 10^{-6} and was found to be about four orders of magnitude larger than the thermal fluctuation variance. Therefore let us call the thermal variance for h_T in (4) d', so that d' is of order 10^{-11} to 10^{-10}. The equilibrium variance for A resulting from (4) is

$$\langle A^2\rangle=\frac{d'}{\beta\tau_0}\ . \tag{5}$$

The largest this quantity can be for the parameter values used earlier is of order 10^{-8}. This is far short of the $\sim 10^{-5}$ we had to use for $\langle A^2(0)\rangle$. This is why the assertion was made that "the magnitude of the initial value fluctuations is much larger than one would expect from thermal fluctuations."

Nevertheless, we see from this result that if the ramping function is chosen instead to be $\epsilon=-\beta'+\beta t$ in which β' is three or four orders of magnitude smaller than β, i.e., the system is just barely subcritical, and if we let the initial value variance $\langle A^2(0)\rangle$, be determined by

$$\tau_0\frac{dA}{dt}=-\beta' A+h_T \tag{6}$$

instead of by (4), then in place of (5) we obtain

$$\langle A^2\rangle=\frac{d'}{\beta'\tau_0}\ , \tag{7}$$

which is now of order 10^{-5} as required. The description of the experiments [2], however, appears to definitely rule out the possibility of such small values for β', which no doubt would be difficult to achieve. We point out this possibility merely to indicate that slightly subcritical relaxation can enhance the apparent size of initial-value thermal fluctuations.

This work was supported by U.S. Air Force Grant No. AFOSR-90-0158, which includes a graduate research stipend.

[1] C. W. Meyer, G. Ahlers, and D. S. Cannell, Phys. Rev. Lett. **59**, 1577 (1987); G. Ahlers, C. W. Meyer, and D. S. Cannell, J. Stat. Phys. **54**, 1121 (1989).
[2] C. W. Meyer, G. Ahlers, and D. S. Cannell, Phys. Rev. A **44**, 2514 (1991).
[3] G. Ahlers, M. C. Cross, P. C. Hohenberg, and S. Safran, J. Fluid Mech. **110**, 297 (1981); J. Swift, P. C. Hohenberg, and G. Ahlers, Phys. Rev. A **43**, 6572 (1991).
[4] H. van Beijeren and E. G. D. Cohen, Phys. Rev. Lett. **60**, 1208 (1988).
[5] Mikhail I. Rabinovich (private communication).
[6] G. Ahlers (private communication).