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Stochastic Calculus in Physics

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The relationship of the Ito–Stratonovich stochastic calculus to studies of weakly colored noise is explained. A functional calculus approach is used to obtain an effective Fokker–Planck equation for the weakly colored noise regime. In a smooth limit, this representation produces the Stratonovich version of the Ito–Stratonovich calculus for white noise. It also provides an approach to steady state behavior for strongly colored noise. Numerical simulation algorithms are explored, and a novel suggestion is made for efficient and accurate simulation of white noise equations.

KEY WORDS: Fokker–Planck equation; functional calculus; colored noise.

1. INTRODUCTION

In recent years the domain of application for stochastic calculus in physics has grown considerably. Initially confined to the description of Borwnian motion and diffusion, stochastic processes are now considered an integral part of a complete description in such diverse areas as hydrodynamics, spectroscopy, cosmology, supersymmetry, and quantum optics, to name just a few specific cases. As stochastic thinking has entered each new arena, the technical details associated with the Ito and Stratonovich versions of the stochastic integral have sometimes given rise to periods of debate and confusion. This feature regarding the applicability of the stochastic calculus has been accentuated by the rapidly increasing use of numerical simulation as a tool for the study of stochastic processes.

The difficulties that arise are a consequence of a mathematical idealization of physical reality: the notion of "white" noise, or in other words, the Markov process. A mathematically precise notion of white noise arose in Wiener's theory[1] of Brownian motion, but the technical dif-

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ficulties were already foreshadowed in Einstein's earlier work[2] on the same subject. Both of these approaches are directly concerned with the process of diffusion. A dynamical underpinning for diffusion was initiated by Langevin[3] and greatly elaborated by Uhlenbeck and Ornstein.[4] This approach eliminated the technical difficulties at the diffusion level of description, but replaced them by identical difficulties at the new level of description. It was Doob[5] who confronted this problem first, and the resolution for white noise was given its rigorous mathematical formulation, the Ito calculus, by Ito[6] shortly thereafter. While this resolved the technical difficulties with white noise descriptions, it introduced an unfamiliar and new type of integration, the stochastic integral. This stochastic integral does not obey the ordinary rules for integrals, and this is the feature of the stochastic calculus that still generates confusion.

An alternative attitude regarding this situation was initiated by Stratonovich.[7] Rather than focusing exclusively on white noise, he suggested that stochastic processes in physics are really non-Markovian; that is, they involve "colored" noise. One then studies "weakly colored" noise and in the end looks at the limit of white noise. This approach leads to an alternative stochastic calculus, the Stratonovich version of the Ito calculus. Its advantage is that the Stratonovich stochastic integral obeys all of the rules of the ordinary integral. Its disadvantage appears to be of a purely mathematical character and has to do with rigorous justification of its rules.

In this paper I will elaborate on the Stratonovich approach to stochastic calculus in physics. A functional calculus approach[8] will be introduced which gives rise to an effective Fokker–Planck equation for weakly colored noise, thereby permitting a potentially rigorous account of the white noise limit. Two additional advantages accrue: (1) steady state behavior for strongly colored noise is also treated; and (2) a greatly improved algorithm for numerical simulations of stochastic equations is justified. These features, together with the new justification of the Stratonovich stochastic calculus, greatly support the functional calculus approach to stochastic calculus in physics.

Section 2 provides an account of the Einstein–Wiener theory of diffusion and its inherent difficulty. This is followed by Section 3 on the Langevin equation and the Doob–Ito resolution. In Section 4, the functional calculus is used to derive an effective Fokker–Planck equation for weakly colored noise, and the Stratonovich version of the stochastic calculus is justified. In Section 5, a strongly colored noise problem is treated. Section 6 is dedicated to the problem of numerical simulation of stochastic equations.

About ten years ago when I first wrote about the Ito–Stratonovich dilemma.[9] Mark Kac chided me for doing so. He said that all it did was make my mentor, and his friend, George Uhlenbeck, irritable. It was better not to even mention Doob or Ito, and to recognize that as physicists, we did not need to be concerned with the mathematical technicalities, since they in no way affected the outcome of our computations of physically measurable results. Since that time, the greatly increased interest by physicists in colored noise problems and in their numerical simulations more than justifies a return to this issue. Remarkably, Mark Kac's interest would have been on the physical side of the question rather than on the purely mathematical. In mathematical physics, he seemed to always pursue clarity, simplicity, and computability. I hope I have succeeded in adhering to these goals here.

2. DIFFUSION A LA EINSTEIN AND WIENER

We may think of diffusion in three-dimensional space as arising from the stochastic differential equation

$$\frac{d}{dt}\mathbf{r}(t)=\tilde{\mathbf{f}}(t) \tag{1}$$

in which $\mathbf{r}(t)$ is the position at time t and $\tilde{\mathbf{f}}(t)$ is a stationary, Gaussian, Markov fluctuating "force" satisfying

$$\langle \tilde{\mathbf{f}}(t)\rangle = 0 \tag{2}$$

$$\langle \tilde{f}_i(t)\,\tilde{f}_j(s)\rangle = 2D\,\delta(t-s)\,\delta_{ij} \tag{3}$$

in which D is the diffusion constant. $\tilde{\mathbf{f}}(t)$ is called "white" noise because of the Dirac delta function in the correlation formula (3). It is straightforward to prove[9] that the conditional probability distribution engendered by Eqs. (1)–(3) is

$$P_2(\mathbf{r}_1 t_1;\mathbf{r}_2 t_2)=[4\pi D(t_2-t_1)]^{-3/2}\exp\left[-\frac{|\mathbf{r}_2-\mathbf{r}_1|^2}{4D(t_2-t_1)}\right] \tag{4}$$

This probability density implies that if a diffusing particle is at $\mathbf{r}_1$ at time t_1, then the probability at time $t_2>t_1$ that its x component is between x_2 and x_2+dx_2, its y component is between y_2 and y_2+dy_2, and its z component is between z_2 and z_2+dz_2 is given by $P_2(\mathbf{r}_1 t_1;\mathbf{r}_2 t_2)\,dx_2\,dy_2\,dz_2$.

Einstein[2] noted in 1906 that the average velocity of change for a component of $\mathbf{r}$ is determined by (4) to be

$$\frac{\langle [r_i(s+t)-r_i(s)][r_i(s+t)-r_i(s)]\rangle^{1/2}}{t}\xrightarrow[t\to 0]{}\frac{(2D)^{1/2}}{t^{1/2}} \tag{5}$$

He noted that this expression:

> becomes infinitely great for an indefinitely small interval of time t; which is evidently impossible, since in that case each suspended particle would move with an infinitely great instantaneous velocity. The reason is that we have implicitly assumed in our development that the events during the time t are to be looked upon as phenomena independent of the events in the time immediately preceding. But this assumption becomes harder to justify the smaller the time t is chosen.

Wiener[1] also realized this difficulty and even proved that $\mathbf{r}(t)$ is nowhere differentiable. This, of course, renders the meaning of (1) doubtful.

3. THE LANGEVIN EQUATION Á LA DOOB AND ITO

The Langevin equation[3,4,9] describes the time evolution of a Brownian particle's velocity rather than directly describing the position. The equation is written

$$M \frac{d}{dt} \mathbf{u}(t) = -\alpha \mathbf{u}(t) + \tilde{\mathbf{F}}(t) \tag{6}$$

in which M is the Brownian particle mass, $\mathbf{u}(t)$ is its velocity, α is the damping parameter, and $\tilde{\mathbf{F}}(t)$ is a stationary, Gaussian, Markov force with stochastic properties

$$\langle \tilde{\mathbf{F}}(t) \rangle = 0 \tag{7}$$

$$\langle \tilde{F}_i(t)\, \tilde{F}_j(s) \rangle = 2k_B T \alpha \delta(t-s)\, \delta_{ij} \tag{8}$$

For a Brownian sphere of radius R in a fluid of viscosity η we have the Stokes formula: $\alpha = 6\pi\eta R$. The Dirac delta function in (8) means that the stochastic force is white noise. In this context, we get the position by integration:

$$\begin{aligned} \mathbf{r}(t) &= \mathbf{r}(0) + \int_0^t ds\, \mathbf{u}(s) \\ &= \mathbf{r}(0) + (M/\alpha)\{1 - \exp[-(\alpha/M)t]\}\, \mathbf{u}(0) \\ &\quad + (1/M) \int_0^t ds \int_0^s ds' \exp[-(\alpha/M)(s-s')]\, \tilde{\mathbf{F}}(s') \end{aligned} \tag{9}$$

If we use $\langle \cdots \rangle$ to denote averaging over $\tilde{\mathbf{F}}$ and $\{\cdots\}$ to denote averaging over the initial velocity $\mathbf{u}(0)$ with respect to a Maxwell distribution, then we obtain

$$\{\langle r_i(t)\, r_j(t) \rangle\} = 2 \frac{k_B T}{\alpha} \left[1 - \frac{M}{\alpha} + \frac{M}{\alpha} \exp\left(-\frac{\alpha}{M} t \right) \right] \delta_{ij} + r_i(0)\, r_j(0) \tag{10}$$

This expression has quite a different t behavior for large t ($\gg M/\alpha$) and for small t ($\ll M/\alpha$). In the former case it goes like t, whereas in the latter case it goes like t^2. Consequently, the analogue to (5) becomes

$$\frac{\langle [r_i(s+t)-r_i(s)][r_i(s+t)-r_i(s)]\rangle^{1/2}}{t} \xrightarrow[t\to 0]{} \frac{k_B T}{M} \tag{11}$$

which it perfectly well behaved. Thus, $\mathbf{r}(t)$ is now differentiable.

We can go further and note from (9) that (1) can be replaced by

$$\frac{d}{dt}\mathbf{r}(t)=\mathbf{u}(t) \tag{12}$$

where $\mathbf{u}(t)$ has the following stochastic properties derivable from (6)–(8):

$$\{\langle \mathbf{u}(t)\rangle\}=0 \tag{13}$$

$$\{\langle u_i(t)\, u_j(t)\rangle\}=(k_B T/M)\exp[-(\alpha/M)\,|t-s|]\,\delta_{ij} \tag{14}$$

Equation (12) is driven by "colored" noise, because the driving "force" $\mathbf{u}(t)$ does not have a Dirac delta function correlation, but instead possesses a non-Markovian memory in a damped exponential correlation with correlation time M/α.

It appears as though we have resolved the Einstein–Wiener dilemma with Langevin's equation. Nevertheless, additional computation shows that

$$\frac{\{\langle [u_i(s+t)-u_i(s)][u_i(s+t)-u_i(s)]\rangle\}^{1/2}}{t} \xrightarrow[t\to 0]{} \frac{k_B T\alpha}{M^2}\frac{1}{t^{1/2}} \tag{15}$$

which clearly diverges. Thus, while the velocity is okay [in the sense of (11)], the acceleration does not exist! We have simply removed the original problem of the differentiability of $\mathbf{r}(t)$ to the nondifferentiability of $\mathbf{u}(t)$. This fact prompted Doob[5] to initiate a reformulation of stochastic differential equations. In his landmark paper of 1942 he said:

> The purpose of the present paper is to apply the methods and results of modern probability theory to the analysis of the Ornstein–Uhlenbeck distribution, its properties and its derivation. It will be seen that the use of rigorous methods actually simplifies some of the formal work, besides clarifying the hypotheses. A stochastic differential equation will be introduced in a rigorous way to give a precise meaning to the Langevin differential equation for the velocity function $(d/dt)\,x(s)$. This will avoid the usual embarassing situation in which the Langevin equation, involving the second derivative of $x(s)$ is used to find a solution $x(s)$ not having a second derivative.

I think this pinpoints the source of Uhlenbeck's irritability!

With this lead, Ito[6] ultimately formulated the Ito stochastic differential equation, with which Langevin's equation takes the form

$$M\, d\mathbf{u}(t) = -\alpha\mathbf{u}(t)\, dt + M\, d\tilde{\mathbf{B}}(t) \tag{16}$$

A key feature of this equation is that $M\, d\tilde{\mathbf{B}}(t)$ cannot be replaced by $\tilde{\mathbf{F}}(t)\, dt$. Said another way, the mean-square of $d\tilde{\mathbf{B}}(t)$ is proportional to dt [not $(dt)^2$] rather than $d\tilde{\mathbf{B}}(t)$. It is this novel dependence on dt in stochastic equations that leads to unusual dependence on step size in numerical algorithms for noise problems, as we will see later.

4. FUNCTIONAL CALCULUS APPROACH

In this section, it is shown how functional calculus may be used to analyze stochastic differential equations. This technique is indifferent to whether white noise or colored noise is involved, at least up to a certain stage of representation. In the white noise context, the Langevin equation provides one representation of the stochastic process, whereas an equivalent representation of all statistical information about the process is also provided by the Fokker–Planck equation.[10,11] Until recently, it was understood that a Fokker–Planck equation exists for Markov processes only. However, for weakly colored noise (non-Markovian) it is possible to derive[8] an effective Fokker–Planck equation as well. This does not transform a non-Markovian process into a Markovian one. Instead, it says that there exists a Markov process (described by the *effective* Fokker–Planck equation) with statistical properties as close as one wants to those of the weakly non-Markovian process. Moreover, in certain special situations, such as steady states, there is even an effective Fokker–Planck equation for strongly colored noise, as is discussed in Section 5. These facts are most transparently exhibited using the functional calculus.

For simplicity of presentation, consider the stochastic differential equation in only one variable, x:

$$\frac{d}{dt}x = W(x) + g(x)\,\tilde{f}(t) \tag{17}$$

in which $W(x)$ and $g(x)$ may be nonlinear functions of x. When $g(x) = 1$, the process is "additive"; otherwise, it is "multiplicative."[9] The noise function $\tilde{f}(t)$ is assumed to be Gaussian and may be either white or colored. If we write

$$\langle \tilde{f}(t)\,\tilde{f}(s) \rangle = C(t-s) \tag{18}$$

then the special choice

$$C_\tau(t-s) = (D/\tau)\exp(-|t-s|/\tau) \tag{19}$$

permits us to cover both cases, since

$$\lim_{\tau \to 0} C_\tau(t-s) = 2D\,\delta(t-s) \tag{20}$$

The Gaussian character of $\mathcal{f}(t)$ is expressed in the functional calculus by a probability distribution functional

$$P[\mathcal{f}] = N \exp\left[-\frac{1}{2}\int ds \int ds'\, \mathcal{f}(s)\,\mathcal{f}(s')\,K(s-s')\right] \tag{21}$$

in which K is the inverse of the $\mathcal{f}$ correlation function C, and N is the normalization expressed by a Feynman–Kac–Wiener path integral over $\mathcal{f}$:

$$N^{-1} = \iint \mathscr{D}\mathcal{f} \exp\left[-\frac{1}{2}\int ds \int ds'\, \mathcal{f}(s)\,\mathcal{f}(s')\,K(s-s')\right] \tag{22}$$

The Feynman–Kac–Wiener path integral is also used to define the probability distribution functional for $x(t)$, the solution to (17). This quantity is

$$P(y, t) = \iint \mathscr{D}\mathcal{f}\, P[\mathcal{f}]\,\delta(y - x(t)) \tag{23}$$

Elsewhere,[8] I have elaborated on this theme and have shown that $P(y, t)$ satisfies the equation

$$\begin{aligned} \frac{\partial}{\partial t} P = & -\frac{\partial}{\partial y}[W(y)P] + \frac{\partial}{\partial y} g(y) \frac{\partial}{\partial y} g(y) \\ & \times \int_0^t ds'\, C(t-s') \iint \mathscr{D}\mathcal{f}\, P[\mathcal{f}]\,\delta(y-x(t)) \\ & \times \exp\left\{\int_{s'}^t ds \left[W'(x(s)) - \frac{g'(x(s))}{g(x(s))} W(x(s))\right]\right\} \end{aligned} \tag{24}$$

in which W' and g' denote the derivatives with respect to x of W and g, respectively. This is an exact equation, which has been given a particularly useful form. It is not a Fokker–Planck equation because something more complicated than just $P(y, t)$ appears in the "diffusion" term (the second piece of the right-hand side). However, when $C(t-s)$ corresponds to a white noise correlation, such as in (20), Eq. (24) becomes the bona fide Fokker–Planck equation

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y} [W(y)P] + D \frac{\partial}{\partial y} g(y) \frac{\partial}{\partial y} g(y)P$$
$$= -\frac{\partial}{\partial y} \{[W(y) + Dg'(y)\, g(y)]P\} + D \frac{\partial^2}{\partial y^2} g^2(y)P \tag{25}$$

Note that these two equivalent forms of the equation are only identical for additive noise.

In the weakly colored noise regime, an effective Fokker–Planck equation also exists. To be explicit, consider the correlation in (19). The diffusion term in (24) can be treated as follows:

$$\int_0^t ds' \frac{D}{\tau} \exp\left(-\frac{t-s'}{\tau}\right) \iint \mathscr{D}f\, P[f]\, \delta(y - x(t))$$
$$\times \exp\left\{\int_{s'}^t ds \left[W'(x(s)) - \frac{g'(x(s))}{g(x(s))} W(x(s))\right]\right\}$$
$$= D \int_0^{t/\tau} d\theta\, e^{-\theta} \iint \mathscr{D}f\, P[f]\, \delta(y - x(s))$$
$$\times \exp\left\{\int_{t-\tau\theta}^t ds \left[W'(x(s)) - \frac{g'(x(s))}{g(x(s))} W(x(s))\right]\right\} \tag{26}$$

where the change of integration from s' to $\theta = (t - s')/\tau$ has been introduced. Up to this point the expression is still exact. "Weakly colored noise" means that it is nearly white, i.e., $\tau \to 0$. I have argued[8] that: *uniformly in* y, τ may be taken sufficiently small such that the following approximation is justified:

$$\int_0^{t/\tau} d\theta\, e^{-\theta} \exp\left\{\int_{t-\tau\theta}^t ds \left[W'(x(s)) - \frac{g'(x(s))}{g(x(s))} W(x(s))\right]\right\}$$
$$\xrightarrow[\tau \to 0]{} \int_0^\infty d\theta\, e^{-\theta} \exp\left\{\tau\theta \left[W'(x(t)) - \frac{g'(x(t))}{g(x(t))} W(x(t))\right]\right\}$$
$$= \left\{1 - \tau \left[W'(x(t)) - \frac{g'(x(t))}{g(x(t))} W(x(t))\right]\right\}^{-1} \tag{27}$$

When this is inserted into the right-hand side of (26), where the $\delta(y - x(t))$

factor can work, and then back into (24), the effective Fokker–Planck equation results:

$$\frac{\partial}{\partial t} P = -\frac{\partial}{\partial y} [W(y)P] + D \frac{\partial}{\partial y} g(y) \frac{\partial}{\partial y} g(y)$$
$$\times \left\{1 - \tau \left[W'(y) - \frac{g'(y)}{g(y)} W(y) \right] \right\}^{-1} P$$
$$= -\frac{\partial}{\partial y} \left[\left(W(y) + Dg'(y)\, g(y) \left\{ 1 - \tau \left[W'(y) - \frac{g'(y)}{g(y)} W(y) \right] \right\}^{-1} \right) P \right]$$
$$+ D \frac{y^2}{\partial y^2} g^2(y) \left\{ 1 - \tau \left[W'(y) - \frac{g'(y)}{g(y)} W(y) \right] \right\}^{-1} P \tag{28}$$

Validity of the last step in (27) requires

$$1 - \tau \left[W'(y) - \frac{g'(y)}{g(y)} W(y) \right] > 0 \qquad \text{for all} \quad y \tag{29}$$

For explicit choices of W and g, this requirement imposes particular constraints on τ. The argument outlined above would benefit from rigorous accessments of the error accrued in time t from use of Eq. (28) to describe the exact behavior in (24).

As long as $\tau > 0$, no special stochastic calculus is required. It is clear from (25) and (28) that as $\tau \to 0$ the effective Fokker–Planck equation goes over smoothly into the white noise limit. This limit is precisely the Stratonovich version of the Ito calculus.(12) I think that if this approach could be made as mathematically rigorous as one would like, then it would simultaneously provide a rigorous underpinning for the Stratonovich version of the Ito calculus, and extend stochastic analysis into the weakly non-Markovian regime. The first benefit would overcome the difficulty with the Stratonovich version of the stochastic calculus, which is that it does not generate a martingale,(12) whereas the second benefit corresponds more closely with real physics, as Einstein already noted.

5. STRONGLY COLORED NOISE AT STEADY STATES

At steady states, it is possible to approximate the exact expression (24) in a way different from the way derived for weakly colored noise. In fact, this steady state approximation appears to work for strongly colored noise as well. By "strongly colored noise" we mean that the τ in (19) is large, and our statement is independent of the magnitude of D. For the approximation described below, we will also surely need to keep D relatively small.

The approximation is to replace the $x(s)$ in the exponential function in (24) by its steady state value, denoted by x_s:

$$\exp\left\{\int_{s'}^{t} ds\left[W'(x(s))-\frac{g'(x(s))}{g(x(s))}W(x(s))\right]\right\}$$
$$\sim\exp\left\{(t-s')\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]\right\} \tag{30}$$

This is reasonable only if D is not too big. Substituting (30) into (26)–(27) yields the steady-state effective Fokker–Planck equation

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial y}[W(y)P]+D\left\{1-\tau\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]\right\}^{-1}$$
$$\times\frac{\partial}{\partial y}g(y)\frac{\partial}{\partial y}g(y)P \tag{31}$$

which may be viewed as a white noise consequence with renormalized diffusion constant

$$D'=D\left\{1-\tau\left[W'(x_s)-\frac{g'(x_s)}{g(x_s)}W(x_s)\right]\right\}^{-1} \tag{32}$$

This approximation was originally proposed by Hanggi *et al.*[13] We see here how easily it sits in the functional calculus representation.

Fox and Roy[14] recently tested this approximation by using it to fit measurements of steady-state dye laser intensity fluctuations obtained by Lett *et al.*[15] The fits of formulas derived from the approximate equation (31), as applied to a fluctuating laser intensity equation, were remarkably good, and placed τ in the strongly colored noise regime. Fitting without using the renormalization diffusion constant D' could not come close to the high quality of the fit with it.

6. NUMERICAL SIMULATIONS

Algorithms for numerical simulation of stochastic differential equations exist for both white noise and colored noise. For additive white noise problems, these algorithms are well tested for accuracy.[16,17] For additive colored noise problems, straightforward extensions exist.[11,18] For the multiplicative noise case, however, the results have not been tested for accuracy until recently. Fox and Roy[19] have investigated the Kubo oscillator problem in which the complex amplitude $a(t)$ satisfies the equation

$$\frac{d}{dt} a(t) = i[\omega_0 + \tilde{\omega}(t)]\, a(t) \tag{33}$$

in which $\tilde{\omega}(t)$ is a real, stochastic frequency with zero mean. We take it as satisfying the equation

$$\frac{d}{dt} \tilde{\omega}(t) = -\lambda \tilde{\omega}(t) + \lambda \tilde{\xi}(t) \tag{34}$$

in which $\tilde{\xi}(t)$ is white noise with zero mean and covariance

$$\langle \tilde{\xi}(t)\, \tilde{\xi}(s) \rangle = Q\delta(t-s) \tag{35}$$

This makes $\tilde{\omega}(t)$ colored noise with covariance

$$\langle \tilde{\omega}(t)\, \tilde{\omega}(s) \rangle = \tfrac{1}{2} Q\lambda e^{-\lambda |t-s|} \tag{36}$$

In the limit $\lambda \to \infty$, $\tilde{\omega}(t)$ becomes white noise.

Equation (33) possesses an associated "Fokker–Planck" equation, which may be solved in closed form, even for colored noise. If we write $a = re^{i\phi}$, we find that r is a constant for the dynamics in (33), but ϕ is described by the probability distribution equation

$$\frac{\partial}{\partial t} P(\phi, t) = \left[-\omega_0 \frac{\partial}{\partial \phi} + D(t) \frac{\partial^2}{\partial \phi^2} \right] P(\phi, t) \tag{37}$$

in which $D(t)$ is defined by

$$D(t) = \tfrac{1}{2} Q(1 - e^{-\lambda t}) \tag{38}$$

If $D(t)$ were a constant $(Q/2)$, then Eq. (37) would indeed be a bona fide Fokker–Planck equation for a Markov process. As it is, however, it is an exact equation for the probability distribution $P(\phi, t)$ in the colored noise regime, but is no longer truly a Fokker–Planck equation. Its solution is just

$$P(\phi, t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \exp\left[im(\phi - \phi_0) - im\omega_0 t - m^2 \int_0^t ds\, D(s) \right] \tag{39}$$

from which all the moments of ϕ may be derived in close form. We have compared these exact expressions for the first three moments of ϕ with the results of numerical simulations.

When we use an algorithm for the coupled system of equations (33) and (34) in the colored noise regime, the additive white noise in (34) is treated by the Box–Mueller algorithm[20] and Eq. (33) is treated by a standard, nonstochastic algorithm. On the other hand, for the white noise limit [$\lambda \to \infty$ in (34)], Eq. (33) may be treated directly with a multiplicative white noise algorithm.[11,18] What we find is that our first approach works extremely well, and the weakly colored noise regime ($\lambda = 10$) reproduces results for the white noise limit very well. The second approach, the direct one, also works well for ϕ, but creates a spurious decay of r, a supposed constant. This artifact can be traced to the dependence on step size of stochastic terms in the algorithms. If step size is denoted by Δ, then a white noise factor goes like $\sqrt{\Delta}$, whereas an ordinary term will involve Δ. For small Δ, $\sqrt{\Delta} \gg \Delta$, and thus larger errors are created in the direct algorithm than in the weakly colored, coupled equation algorithm if Δ is the same for both.

Our suggestion is to use the coupled equation algorithm for weakly colored noise instead of the direct algorithm for white noise. It is surprising that even when λ is not very big ($\lambda \sim 10$), the weakly colored noise is effectively white. This adds weight to the view expressed earlier in this paper that physical reality calls for weakly colored noise rather than for white noise. Here, it greatly simplifies the numerical algorithm and improves accuracy. Finally, using the functional calculus, it appears to be possible to establish a mathematically rigorous basis for the Stratonovich stochastic calculus in the weakly colored noise regime.

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