# Effective equilibrium states in mixtures of active particles driven by colored noise

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We consider the steady-state behavior of pairs of active particles having different persistence times and diffusivities. To this purpose we employ the active Ornstein-Uhlenbeck model, where the particles are driven by colored noises with exponential correlation functions whose intensities and correlation times vary from species to species. By extending Fox's theory to many components, we derive by functional calculus an approximate Fokker-Planck equation for the configurational distribution function of the system. After illustrating the predicted distribution in the solvable case of two particles interacting via a harmonic potential, we consider systems of particles repelling through inverse power laws potentials. We compare the analytic predictions to computer simulations for such soft-repulsive interactions in one dimension, and show that at linear order in the persistence times the theory is satisfactory.

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## I. INTRODUCTION

The study of active particles has recently attracted rapidly increasing attention of scientists belonging to different disciplines due to the current interest in the physical principles governing the behavior of fish schools, herds of animals, bacteria, collections of cells and/or man-made active colloids [1–3]. All these systems in order to move convert energy via metabolic or chemical reactions and are thus out of equilibrium [4]. On the theory side, many fundamental aspects of active systems can be described with some minimalistic models based on spherical particles [5, 6]. Even in the absence of attractive interactions such particles can exhibit intriguing individual and collective behavior, induced solely by their persistent motion, such as the accumulation at the system boundary [7, 8], the separation into a dilute and a dense phase [9, 10] and wetting or capillary condensation condensation transitions [8, 11].

While the majority of studies are concerned with systems whose constituents are all identical, in real situations it is common to observe assemblies of active particles of different nature. The obvious question is how does the heterogeneity affect the collective behavior of such mixtures [3]. For example, doping a passive fluid with a small number of active particles significantly alters its structural and dynamical properties by supporting the formation of clusters [12] and, at higher densities, crystallization [12, 13]. On the other hand, active dopants with a short persistence length were reported to aggregate in cages [14]. On immersing large colloids into a bath of smaller active colloids, the former effectively are rendered active, which becomes manifest through activity-enhanced diffusivities [15, 16] and depletion forces [8, 17–19]. Employing shape-anisotropic colloids [19–21] or manufacturing activity gradients [22], these effects can be used to generate directed motion of the colloids. Moreover, the segregation between two passive species has been reported as the result of coupling only one species to the active bath [23].

The mixtures described so far consist of species which differ in their shape (interaction potential) and particle number. Seeking for the closest analogy to the motility-induced phenomena observed for a single active species, we are particularly interested in particles solely distinguished by a difference in their activity. The most intriguing feature of such multicomponent active systems is their capability to demix, which cannot be attributed to the physical mechanisms also present in equilibrium mixtures. Some recent investigations have focused on such a binary mixture of an active and a passive species [24–26] or particles with different finite activities [26–29]. Quite intuitively, active phase separation phenomena can be described using the concept of an effective temperature, enhanced by activity [30], which has also been applied to mixtures [27, 31, 32]. Relatedly, the demixing of particles with the same mobility but different diffusion coefficients has been recently reported [33].

The model of active particles propelled by so-called Ornstein-Uhlenbeck processes (OUPs) provides a convenient starting point of many theoretical studies, since their equations of motion do not resolve the orientational degrees of freedom. A minimalistic strategy is based on the multidimensional generalizations of the unified colored noise approximation (UCNA) [34, 35] or a similar

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approach by Fox [36, 37], see Ref. 38 for a detailed comparison. This procedure yields an approximate Smoluchowski equation which, in the steady state, admits an analytic solution for the configurational probability distribution [39], and closed formulas for active pressure and interfacial tension [40–42]. The former allows to define effective interaction potentials, which can be directly used to determine density profiles [38, 43, 44] and rate equations [45] of individual ideal particles and, when implemented in equilibrium liquid-state theory, the structure and phase behavior of interacting systems [11, 46, 47]. The described effective equilibrium approach is most accurate in one spatial dimension and for small persistence time, which can be explicitly verified by studying exactly solvable models [42].

In the present work we take our effective equilibrium model to the next level by further generalizing the Fox approach, which turns out to be more promising, to mixtures of different active particles. The paper is organized as follows. The description of the model and the generalization of the Fox approach are presented in Sec. II. We then verify in Sec. III the accuracy of the theory at linear order in the persistence times by studying an exactly solvable harmonic problem. In Sec. IV we discuss the configurational probability distribution and compare to numerical results for two active OUPs. Finally, we draw some conclusions in Sec. V.

## II. THEORY

In order to study a system of $N$ active particles in $\mathfrak{d}$ spatial dimensions having species dependent diffusivities $D_{\mathrm{a}}^{(i)}$ and persistence times $\tau_{\mathrm{a}}^{(i)}$ where the index $i \in \{1 \dots N\}$, we generalize the microscopic one-component active OUPs model of Refs. 38, 39, and 43 to the case where different types of Gaussian stochastic driving terms are present. To do so, we introduce a component-wise notation (compare, e.g., Ref. 43) for $\mathfrak{d}N$-dimensional arrays $x_\alpha(t)$ denoting the coordinates of the particles evolving according to

$$\dot{x}_\alpha(t) = D_{\mathrm{t}}\beta F_\alpha(x_1, x_2, \dots, x_{\mathfrak{d}N}) + \chi_\alpha(t)\,, \tag{1}$$

where $\alpha \in \{1 \dots \mathfrak{d}N\}$, $F_\alpha$ is a conservative force due to passive interactions, $D_{\mathrm{t}}\beta$ is the inverse friction coefficient (related to the translational diffusivity $D_{\mathrm{t}}$ in a Brownian system).

The Gaussian stochastic noise $\chi_\alpha(t)$ evolves in time according to

$$\dot{\chi}_\alpha(t) = -\frac{1}{\tau_{\mathrm{a}}^{(\alpha)}}\chi_\alpha(t) + \frac{\sqrt{\mathcal{D}_{\mathrm{a}}^{(\alpha)}}}{\tau_{\mathrm{a}}^{(\alpha)}}\,\xi_\alpha(t)\,, \tag{2}$$

with the white noise $\xi_\alpha(t)$, which has the time correlator $\langle \xi_\alpha(t)\xi_\beta(t')\rangle = 2D_{\mathrm{t}}\delta_{\alpha\beta}\delta(t-t')$, and $\mathcal{D}_{\mathrm{a}}^{(\alpha)} = D_{\mathrm{a}}^{(\alpha)}/D_{\mathrm{t}}$. It has zero average and the tensorial time correlator

$$C_{\alpha\beta}(t-t') := \langle \chi_\alpha(t)\chi_\beta(t')\rangle = \frac{D_{\mathrm{a}}^{(\alpha)}}{\tau_{\mathrm{a}}^{(\alpha)}}\delta_{\alpha\beta}e^{-\frac{|t-t'|}{\tau_{\mathrm{a}}^{(\alpha)}}}\,. \tag{3}$$

The probability distribution functional of $\chi_\alpha(t)$ has the Gaussian representation:

$$P_N[\{\chi_\alpha\}] \propto \exp\left(-\frac{1}{2}\iint \mathrm{d}s\,\mathrm{d}s' \sum_{\alpha\beta}\chi_\alpha(s)K_{\alpha\beta}(s-s')\chi_\beta(s')\right) \tag{4}$$

and is equipped with a tensorial kernel $K_{\alpha\beta}(t-t')$, the inverse of $C_{\alpha\beta}$.

In the appendix, by extending Fox's approximation to an arbitrary number of (active) components, we show that the configurational distribution of positions of particles

$$f_N(\{y_\alpha\}, t) = \int D[\{\chi_\alpha\}]P_N[\{\chi_\alpha\}] \prod_\alpha \delta(y_\alpha - x_\alpha(t)) \tag{5}$$

evolves according to the following Fokker-Planck equation:

$$\frac{\partial f_N(\{y_\alpha\},t)}{\partial t} = -\sum_\beta \frac{\partial}{\partial y_\beta}\Big(D_{\mathrm{t}}\beta F_\beta(\{y_\alpha\})f_N(\{y_\alpha\},t) - D_{\mathrm{a}}^{(\beta)}\sum_\gamma \frac{\partial}{\partial y_\gamma} f_N(\{y_\alpha\},t)\Gamma^{-1}_{\gamma\beta}(\{y_\alpha\})\Big) \tag{6}$$

with friction tensor

$$\Gamma_{\gamma\beta} = \delta_{\gamma\beta} - \tau_{\mathrm{a}}^{(\beta)}D_{\mathrm{t}}\beta\frac{\partial F_\gamma}{\partial x_\beta}\,. \tag{7}$$

Switching to Cartesian coordinates we find the following Fokker-Planck equation:

$$\frac{\partial f_N(\mathbf{r}^N,t)}{\partial t} = -D_{\mathrm{t}}\sum_i \boldsymbol{\nabla}_i \cdot \Big(\beta\mathbf{F}_i(\mathbf{r}^N)f_N(\mathbf{r}^N,t) - \sum_j \boldsymbol{\nabla}_j \cdot \big(\mathcal{D}_{ji}(\mathbf{r}^N)f_N(\mathbf{r}^N,t)\big)\Big)\,, \tag{8}$$

where the non dimensional diffusion tensor $\mathcal{D}_{ji}$ is defined as $\mathcal{D}_{ji}(\mathbf{r}^N) = \mathcal{D}_{\mathrm{a}}^{(i)}\,\Gamma^{-1}_{ji}(\mathbf{r}^N)$. Assuming that the conservative force arises from an interaction potential $u_{ij}$ between each pair of particles and defining

$$\tilde{\tau}^{(\alpha)} = D_{\mathrm{t}}\beta\tau_{\mathrm{a}}^{(\alpha)}\,, \tag{9}$$

the friction tensor from Eq. (7) reads [38, 43]

$$\Gamma_{ij}(\mathbf{r}^N) = \mathbf{1}\delta_{ij} + \tilde{\tau}^{(j)}\boldsymbol{\nabla}_i\boldsymbol{\nabla}_j\sum_{k\neq i}^{N} u_{ik}(\mathbf{r}_i,\mathbf{r}_k)\,. \tag{10}$$

In general, $\mathcal{D}_{ij}(\mathbf{r}^N)$ is not symmetric, i.e., $\mathcal{D}_{ij} \neq \mathcal{D}_{ji}$, if there are at least two different particle species. Regarding the structure of Eq. (8), transposing $\mathcal{D}_{ij}$ changes the components of the probability current but not the overall time evolution of the probability distribution. Further note that $\Gamma_{ij} = \Gamma_{ji}$ only if $i, j$ belong to the same species.

Intriguingly, the generalized Unified Colored Noise Approximation (UCNA) [34, 35] gives rise to a mobility matrix

$$\Gamma^{\rm ucna}_{\gamma\beta} = \delta_{\gamma\beta} - \tau_{\rm a}^{(\gamma)} D_{\rm t}\beta \frac{\partial F_\gamma}{\partial x_\beta} = \Gamma_{\beta\gamma} \tag{11}$$

which is the transpose of the Fox result in Eq. (7). However, we find that the corresponding effective diffusion tensor $\mathcal{D}^{\rm ucna}_{ji} := \mathcal{D}^{(i)}_{\rm a} \, (\Gamma^{\rm ucna})^{-1}_{ji} = \mathcal{D}^{(i)}_{\rm a} \, \Gamma^{-1}_{ij} \neq \mathcal{D}_{ij}$ is totally unrelated to $\mathcal{D}_{ij}$, contrasting the relation in Eq. (11). Therefore, UCNA and Fox do not share the same steady state as it is the case in a one-component system at sufficiently high activity. As we will see below, this is even the case at linear order in the persistence time.

While the steady-state condition requires the vanishing of the divergence of the probability current, the condition of detailed balance involves the vanishing of all components of the probability current so that from Eq. (6) we find:

$$f_N \Big( \beta F_\beta - \mathcal{D}^{(\beta)}_{\rm a} \sum_\gamma \frac{\partial}{\partial y_\gamma} \Gamma^{-1}_{\gamma\beta} \Big) = \mathcal{D}^{(\beta)}_{\rm a} \sum_\gamma \Gamma^{-1}_{\gamma\beta} \frac{\partial}{\partial y_\gamma} f_N \, . \tag{12}$$

In general, the solution of such an equation is not known, but it reduces to a Boltzmann-like distribution in the case of a single species [39].

For completeness, we consider the most general case where some particles are subject to an additional translational Brownian white noise, referred to as thermal noise in the following, which becomes relevant, e.g., for studying a mixture of an active and a passive Brownian species. Generalizing Fox's approximation [45], it is straightforward to rewrite Eq. (8) with the generalized tensor

$$\mathcal{D}_{ji}(\mathbf{r}^N) = \mathbf{1} \delta_{ji} I^{(i)}_{\rm t} + \mathcal{D}^{(i)}_{\rm a} \, \Gamma^{-1}_{ji}(\mathbf{r}^N) \, , \tag{13}$$

where $I^{(i)}_{\rm t}$ takes the values 1 in the presence and 0 in the absence of thermal noise acting on particle $i$. Equation (8) with Eq. (13) represents the full extension of Fox's approximation [36, 37] for one component active fluids [38, 45, 46] to the multicomponent case where the fluid contains different species and each species is subjected to active (colored) noise whose intensity depends on the persistence time and the species.

## III. AN ELEMENTARY TEST

Before, discussing the more realistic case of a mixture, we consider perhaps the simplest model of interacting particles which lends itself to an analytic solution and may serve as a benchmark for our theory. An elastic dimer in a $\mathfrak{d} = 1$ dimensional well and subject to two different colored Gaussian baths is represented by two particles mutually coupled by a harmonic potential $u(x_1 - x_2) = \alpha^2 (x_1 - x_2)^2/2$ and each confined by a harmonic external potential $v(x_\alpha) = \omega^2 x_\alpha^2/2$. Their dynamics is described by the evolution equations

$$\dot{x}_\alpha(t) = D_{\rm t}\beta F_\alpha(x_1, x_2) + \chi_\alpha(t) \, , \tag{14}$$

where $F_\alpha = -\frac{\rm d}{{\rm d}x_\alpha}(v(x_\alpha) + u(x_\alpha - x_\beta))$. The friction $1/(D_{\rm t}\beta)$ is the same for both species, but the parameters $\tau_{\rm a}^{(\alpha)}$ and $\mathcal{D}_{\rm a}^{(\alpha)}$ characterizing the stochastic driving term $\chi_\alpha(t)$, defined in Eq. (2), are different. Explicitly we have

$$\dot{x}_1 = -D_{\rm t}\beta(\omega^2 x_1 + \alpha^2(x_1 - x_2)) + \chi_1(t) \, , \tag{15}$$

$$\dot{x}_2 = -D_{\rm t}\beta(\omega^2 x_2 - \alpha^2(x_1 - x_2)) + \chi_2(t) \, . \tag{16}$$

This model has been first used in statistical mechanics in the context of the virial theorem by Riddell and Uhlenbeck [48] and recently by one of us [42] in the framework of active systems.

The model, being linear, can be solved analytically by direct integration of equations of motion. It is convenient to switch to collective variables $q = x_1 - x_2$ and $Q = (x_1 + x_2)/2$ and to the renormalized spring constant $\Omega^2 = \omega^2 + 2\alpha^2$. The steady-state equal-time pair correlations read

$$\langle Q(t)Q(t) \rangle = \frac{1}{4\beta\omega^2} \Big( \frac{\mathcal{D}^{(1)}_{\rm a}}{1 + \tilde{\tau}^{(1)}\omega^2} + \frac{\mathcal{D}^{(2)}_{\rm a}}{1 + \tilde{\tau}^{(2)}\omega^2} \Big) \, , \tag{17}$$

$$\langle q(t)q(t) \rangle = \frac{1}{\beta\Omega^2} \Big( \frac{\mathcal{D}^{(1)}_{\rm a}}{1 + \tilde{\tau}^{(1)}\Omega^2} + \frac{\mathcal{D}^{(2)}_{\rm a}}{1 + \tilde{\tau}^{(2)}\Omega^2} \Big) \, . \tag{18}$$

with $\tilde{\tau}^{(\alpha)}$ defined by (9). Such an exact result will now be used to compare with the generalized Fox approximation.

### A. Fox's approximation for two oscillators

We now compute the averages featuring in Eqs. (17) and (18) using the approximate Fox theory for two components. The evolution of the probability distribution function of the state, $\{y_\alpha\}$, at time $t$ described by the Fokker-Planck equation, Eq. (6), can be written in the compact form

$$\frac{\partial}{\partial t} f_N(\{y_\alpha\}, t) = \mathcal{L}_{\rm FP} f_N(\{y_\alpha\}, t) \tag{19}$$

where $\mathcal{L}_{\rm FP}$ represents the Fokker-Planck operator. It follows that the average of an observable $\mathcal{O}(\{y_\alpha\})$ evolves according to

$$\frac{\partial}{\partial t} \langle \mathcal{O}(\{y_\alpha\}, t) \rangle = \langle \mathcal{L}^\dagger_{\rm FP} \mathcal{O}(\{y_\alpha\}, t) \rangle \tag{20}$$

where $\mathcal{L}^\dagger_{\rm FP}$ is the adjoint operator of the Fokker-Planck operator $\mathcal{L}_{\rm FP}$ and $\langle \mathcal{O}(\{y_\alpha\}, t) \rangle \equiv \int d\{y_\alpha\} f_N(\{y_\alpha\}, t) \mathcal{O}(\{y_\alpha\})$.

For the case of the Riddell-Uhlenbeck model, let us introduce the following symbols to shorten the notation: $D_{\rm t}\beta F_\alpha = \sum_{\beta=1}^{2} M_{\alpha\beta}x_\beta$, $M_{11} = M_{22} = -D_{\rm t}\beta(\omega^2+\alpha^2)$ and $M_{12} = M_{21} = D_{\rm t}\beta\alpha^2$ and write the evolution equations under the form:

$$\frac{d}{dt}\langle x_\alpha x_\beta\rangle = \sum_\gamma [M_{\alpha\gamma}\langle x_\gamma x_\beta\rangle + M_{\beta\gamma}\langle x_\gamma x_\alpha\rangle] + D_{\rm t}[\mathcal{D}_{\rm a}^{(\beta)}\Gamma^{-1}_{\alpha\beta} + \mathcal{D}_{\rm a}^{(\alpha)}\Gamma^{-1}_{\beta\alpha}] \quad (21)$$

with $\Gamma_{\alpha\beta} = \delta_{\alpha\beta} - \tilde{\tau}^{(\beta)}\partial F_\alpha/\partial x_\beta$ according to Eq. (7). Since the motion is confined by the external potential, the chosen observables are limited, and by the ergodic theorem [49] the l.h.s of (21) vanishes as $t\to\infty$ as the system approaches the steady state. We obtain a simple set of linear equations for the correlators $\langle x_\alpha x_\beta\rangle$, which can be easily solved.

Evaluating the inverse matrix $\Gamma^{-1}_{\alpha\beta}$ to first order in $\tilde{\tau}^{(\alpha)}$

$$\begin{aligned}\Gamma^{-1}_{11} &= 1-\tilde{\tau}^{(1)}(\omega^2+\alpha^2)\,,\\ \Gamma^{-1}_{22} &= 1-\tilde{\tau}^{(2)}(\omega^2+\alpha^2)\,,\\ \Gamma^{-1}_{12} &= \tilde{\tau}^{(2)}\alpha^2\,,\\ \Gamma^{-1}_{21} &= \tilde{\tau}^{(1)}\alpha^2\end{aligned} \quad (22)$$

and we obtain the following expression for the equal-time correlations of the collective variables according to Fox's theory:

$$\langle QQ\rangle = \frac{1}{4\beta\omega^2}\Big(\mathcal{D}_{\rm a}^{(1)}(1-\tilde{\tau}^{(1)}\omega^2) + \mathcal{D}_{\rm a}^{(2)}(1-\tilde{\tau}^{(2)}\omega^2)\Big) \quad (23)$$

$$\langle qq\rangle = \frac{1}{\beta\Omega^2}\Big(\mathcal{D}_{\rm a}^{(1)}(1-\tilde{\tau}^{(1)}\Omega^2) + \mathcal{D}_{\rm a}^{(2)}(1-\tilde{\tau}^{(2)}\Omega^2)\Big). \quad (24)$$

By comparing Eqs. (23) and (24) with the corresponding exact expressions (17) and (18) we see that Fox's result agrees to linear order in $\tilde{\tau}^{(\alpha)}$. One can easily see from comparing Eq. (7) and Eq. (11) that UCNA results for the off-diagonal friction matrix elements in Eq. (22) and the correlators in Eq. (24) are different from those found in the present treatment, based on the Fox approximation, which is exact in the small $\tilde{\tau}^{(\alpha)}$ limit. Therefore, we shall not digress to further discuss the multicomponent UCNA equations.

## IV. PROBABILITY DISTRIBUTION FUNCTION

Having seen that a system of two active particles with different diffusivities and (small) persistence times are accurately described by the multicomponent Fox approach, our goal is to describe a system with $\kappa$ components in $\mathfrak{d}$ dimensions. As shown in Sec. II, the effective Fokker-Planck equation (8) does, in general, not admit current-free steady states. We thus write the condition for the existence of a steady-state solution in terms of the components

$$\mathbf{J}_{i_\mu} = -\left(\beta\mathbf{F}_{i_\mu}f_N - \sum_{\nu=1}^{\kappa}\sum_{j_\nu=1}^{N_\nu}\boldsymbol{\nabla}_{j_\nu}\cdot(\mathcal{D}_{j_\nu i_\mu}f_N)\right) \quad (25)$$

of the probability current vector $\mathbf{J}$, where the sum over particles $i$ has been transformed into a double sum over particle species $\mu$ and particles $i_\mu$ of each species, so that $N=\sum_\mu N_\mu$.

The steady-state condition then becomes

$$\sum_{\mu=1}^{\kappa}\sum_{i_\mu=1}^{N_\mu}\boldsymbol{\nabla}_{i_\mu}\cdot\mathbf{J}_{i_\mu} = 0\,, \quad (26)$$

generalizing the expression $\mathbf{J}_i = 0$ for one component [38]. Note that is not possible to write the above equation in a thermodynamical version [41] defining a conservative effective force [38, 46]. The prerequisite to derive an effective force of any kind is the existence of a Boltzmann-like configurational probability distribution $f_N(\mathbf{r}^N)$, which is known analytically for a one-component system if no thermal noise is present [39].

### A. Two-particle current

From now on, we restrict ourselves to $N=2$ particles and silently omit the additional index denoting the species $\upsilon$ and $\mu$ of particle $1\simeq 1_\upsilon$ and of particle $2\simeq 2_\upsilon$ or $2\simeq 1_\mu$ if $\mu\neq\upsilon$. As both the interaction force and the effective diffusion tensor only depend on their distance $r=|\mathbf{r}_1-\mathbf{r}_2|$ we can define a common derivative operator $\boldsymbol{\nabla}\equiv\boldsymbol{\nabla}_1=-\boldsymbol{\nabla}_2$. It then follows from Eq. (25) that

$$\boldsymbol{\nabla}\cdot\mathbf{J}_1 - \boldsymbol{\nabla}\cdot\mathbf{J}_2 = 0\,. \quad (27)$$

Explicitly, the two currents are given by:

$$\begin{aligned}\mathbf{J}_1(r) =& f_2(r)\boldsymbol{\nabla}(-\beta u(r) - \mathcal{D}_{11} + \mathcal{D}_{21})\\ &+ (-\mathcal{D}_{11}+\mathcal{D}_{21})\cdot\boldsymbol{\nabla}f_2\,,\\ \mathbf{J}_2(r) =& f_2(r)\boldsymbol{\nabla}(\beta u(r) - \mathcal{D}_{12} + \mathcal{D}_{22})\\ &+ (-\mathcal{D}_{12}+\mathcal{D}_{22})\cdot\boldsymbol{\nabla}f_2\,.\end{aligned} \quad (28)$$

Consider first a two-body system with a single component $\mu=\upsilon$, where $\mathcal{D}_{11}=\mathcal{D}_{22}$ and $\mathcal{D}_{21}=\mathcal{D}_{12}$ and thus Eq. (27) is trivially fulfilled. In general, the zero-divergence condition (27) is satisfied by $\mathbf{J}_1 = \mathbf{J}_2 + \mathbf{X}$, where $\mathbf{X}$ is a constant in $\mathfrak{d}=1$ dimension or a divergence-free vector field for $\mathfrak{d}=3$.

Let us now consider for the sake of simplicity and illustration purpose the one-dimensional case, where we define $x=x_1-x_2$. The two particles belonging to different species interact with the same finite-range and symmetric pair potential $u(x)$ with $\lim_{x\to\pm\infty}u(x)=0$. By subtracting the second current in Eq. (28) from the first one we obtain

$$X = -2f_2(x)\frac{d}{dx}\Big(\beta u(x) + \mathcal{D}_{\rm m}\Big) - 2\mathcal{D}_{\rm m}\frac{d}{dx}f_2(x) \quad (29)$$

where we defined

$$\mathcal{D}_{\rm m} := \frac{1}{2}\big(\mathcal{D}_{11} - \mathcal{D}_{21} + \mathcal{D}_{22} - \mathcal{D}_{12}\big)\,. \tag{30}$$

Such an inhomogeneous first order linear differential equation can be solved by introducing the so-called integrating factor

$$\psi_2(x) = \exp\Big(-\int_{-\infty}^{x} \frac{{\rm d}s}{\mathcal{D}_{\rm m}(s)}\,\frac{d}{ds}\left(\beta u(s) + \mathcal{D}_{\rm m}(s)\right)\Big), \tag{31}$$

Due to the symmetry of the pair potential the integrating factor has the property $\lim_{x\to\pm\infty}\psi_2(x) = 1$. The general solution of Eq. (29) must be of the form

$$f_2(x) = \psi_2(x)\Big(A - \frac{X}{2}\int_{-\infty}^{x}\frac{{\rm d}s}{\mathcal{D}_{\rm m}(s)\,\psi_2(s)}\Big) \tag{32}$$

so that $\lim_{x\to-\infty} f_2(x) = A$. Now, the value of the distribution function at infinity must be identical to the value at minus infinity ( $\lim_{x\to\infty} f_2(x) = A$ ) and we may conclude that:

$$A = \lim_{x\to\infty}\psi_2(x)\Big(A - \frac{X}{2}\int_{-\infty}^{x}\frac{{\rm d}s}{\mathcal{D}_{\rm m}(s)\,\psi_2(s)}\Big). \tag{33}$$

The only solution is $X=0$, thus the two currents $J_1 = J_2$ must be equal and $f_2(x) = \psi_2(x)$ is given by Eq. (31), i.e., $A=1$. Combining Eq. (29) for $X=0$ with Eq. (28) we find

$$\begin{aligned} J_1 = J_2 = \frac{f_2(x)}{2\mathcal{D}_{\rm m}}\,&\big((\mathcal{D}_{11} - \mathcal{D}_{21} - (\mathcal{D}_{22} - \mathcal{D}_{12}))\boldsymbol{\nabla}\beta u(x)\\ &+ (\mathcal{D}_{11} - \mathcal{D}_{21})\boldsymbol{\nabla}(\mathcal{D}_{22} - \mathcal{D}_{12})\\ &-(\mathcal{D}_{22} - \mathcal{D}_{12})\boldsymbol{\nabla}(\mathcal{D}_{11} - \mathcal{D}_{21})\big)\,. \end{aligned} \tag{34}$$

One can see that $J_1 = J_2 = 0$ only if $\mathcal{D}_{11} - \mathcal{D}_{21} = \mathcal{D}_{22} - \mathcal{D}_{12} = \mathcal{D}_{\rm m}$, a condition which is realized when the two particles are identical.

### B. An example of non-zero partial currents

To illustrate that the equality $J_1 = J_2$ of the (non-vanishing) partial currents established in Sec. IV A does not hold only for particles interacting via soft-repulsive potentials but also for other interactions, we consider of two different active OUPs (without thermal noise) bound by the harmonic pair potential $u(x) = \alpha^2 x^2/2$, cf., Sec. III. In this case, we must have $\lim_{x\to\pm\infty}\psi_2(x) = 0$ for the integrating factor in Eq. (31), because the potential is confining. Therefore, the argumentation leading to Eq. (34) is no longer justified. However, we also know the form of the equal-time pair correlation $\langle q(t)\,q(t)\rangle = \mathcal{D}_{\rm m}/(\beta\alpha^2)$, compare the result at linear order in $\tilde{\tau}^{(\alpha)}$ given by Eq. (24) with $\Omega^2 \equiv 2\alpha^2$. We can thus write the steady-state probability distribution

$$f_2(x) \sim \exp\Big(\frac{-x^2}{2\,\langle q\,q\rangle}\Big) \sim \exp\Big(\frac{-\beta\alpha^2}{2\mathcal{D}_{\rm m}}x^2\Big) \sim \exp\Big(\frac{-\beta u(x)}{\mathcal{D}_{\rm m}}\Big) \tag{35}$$

explicitly as a Gaussian.

It is easily verified that a distribution of the form (35) gives $X = 0$ in Eq. (29), since $\mathcal{D}_{\rm m}$ does not depend on $x$ for the employed potential. We immediately get from Eq. (34) the position-dependent currents

$$\begin{aligned} J_1 = J_2 &= \frac{f_2(x)}{2\mathcal{D}_{\rm m}}(\mathcal{D}_{11} - \mathcal{D}_{21} - \mathcal{D}_{22} + \mathcal{D}_{12})\boldsymbol{\nabla}\beta u(x)\\ &= \alpha^2 x f_2(x)\frac{\big((\tilde{\tau}^{(1)} - \tilde{\tau}^{(2)})(\mathcal{D}_{\rm a}^{(1)} + \mathcal{D}_{\rm a}^{(2)})\big)\alpha^2 - \mathcal{D}_{\rm a}^{(1)} + \mathcal{D}_{\rm a}^{(2)}}{\big((\tilde{\tau}^{(1)} - \tilde{\tau}^{(2)})(\mathcal{D}_{\rm a}^{(1)} - \mathcal{D}_{\rm a}^{(2)})\big)\alpha^2 - \mathcal{D}_{\rm a}^{(1)} - \mathcal{D}_{\rm a}^{(2)}}\,. \end{aligned} \tag{36}$$

From these formulas it is easy to verify that the currents vanish for two identical particles with both $\tau^{(1)} = \tau^{(2)}$ and $\mathcal{D}_{\rm a}^{(1)} = \mathcal{D}_{\rm a}^{(2)}$. The full formula for the probability distribution $f_2(x)$ follows from the consideration in the following section, which hold for an arbitrary bare interaction potential.

### C. Effective potentials

With the knowledge of the two-particle probability distribution $f_2$, as given in one dimension by $f_2(x) \equiv \psi_2(x)$ from Eq. (31), one can define an effective force $\mathbf{F}_i^{\rm eff}$ on particle $i \in \{1,2\}$ from the condition $\beta\mathbf{F}_i^{\rm eff} f_2 - \boldsymbol{\nabla}_i f_2 = 0$. Note that defining an effective force $\beta\tilde{\mathbf{F}}_i^{\rm eff}$ equal to $\mathcal{D}_{1i}^{-1}\cdot\mathbf{J}_1 + \mathcal{D}_{2i}^{-1}\cdot\mathbf{J}_2 + \boldsymbol{\nabla}_i f_2$, which in the one component system is equivalent to the first definition [38], does not yield the same result in the general case of a mixture with non-vanishing currents. We now turn to discuss the effective interaction potential between two members of species $\upsilon$ and $\mu$ in $\mathfrak{d}$ dimensions, obtained from equating the currents in Eq. (28) as

$$\boldsymbol{\nabla}\beta u_{\upsilon\mu}^{\rm eff}(r) = -\boldsymbol{\nabla}\ln f_2 = \mathcal{D}_{\rm m}^{-1}\cdot\boldsymbol{\nabla}\cdot(\mathbf{1}\beta u(r) + \mathcal{D}_{\rm m}) \tag{37}$$

with $\mathcal{D}_{\rm m}$ defined as in Eq. (30).

The effective potentials $u_{\upsilon\upsilon}^{\rm eff}$ between members of the same species $\upsilon$ can be obtained in various alternative ways, e.g., by setting either current in Eq. (28) to 0. The explicit form of the $u_{\upsilon\upsilon}^{\rm eff}$ has been discussed in detail in Ref. 38. Without thermal noise, i.e., setting in Eq. (13) $I_{\rm t}^{(i)} = 0$ for all $i$, the closed analytical expression

$$\beta u_{\upsilon\upsilon}^{\rm eff}(r) = \beta\frac{u(r) + \tilde{\tau}^{(\upsilon)}\left(\frac{\partial u(r)}{\partial r}\right)^2}{\mathcal{D}_{\rm a}^{(\upsilon)}} - \ln\left(\left|E_1^{(\upsilon)}\right|^{(\mathfrak{d}-1)}\left|E_2^{(\upsilon)}\right|\right), \tag{38}$$

where the superscript $(\upsilon)$ is short for $(i_\upsilon)$. The above expression is valid in any spatial dimension $\mathfrak{d}$, where

$$E_n^{(\upsilon)}(r) = 1 + 2\tilde{\tau}^{(\upsilon)} r^{n-2}\frac{\partial^n u(r)}{\partial r^n} \tag{39}$$

with $n \in \{1,2\}$ are the eigenvalues of $\Gamma_{ij}$, defined in Eq. (10), for two particles of species $\upsilon$. In this special case, $E_n^{(\upsilon)}$ are equivalent (up to the factor $\mathcal{D}_{\rm a}^{(\upsilon)}$) to the

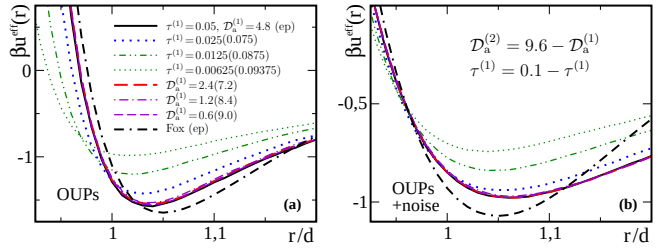


FIG. 1. Effective potentials in one dimension from OUPs simulation and the Fox theory, Eq. (42), between two active particles with the parameters $\tau^{(1)}$ and $\mathcal{D}_{\rm a}^{(1)}$ (as labeled) with $\tau^{(2)} = 0.1 - \tau^{(1)}$ and $\mathcal{D}_{\rm a}^{(2)} = 9.6 - \mathcal{D}_{\rm a}^{(1)}$, such that $\bar{\tau} = 0.05$ and $\bar{\mathcal{D}}_{\rm a} = 4.8$ (the results are invariant under exchanging the labels 1 and 2). Here we consider only particles with at least one pair of equal parameters ($\tau^{(2)} = \tau^{(1)}$ or $\mathcal{D}_{\rm a}^{(2)} = \mathcal{D}_{\rm a}^{(1)}$), such that the results only depend on $|\Delta\mathcal{D}_{\rm a}|$ or $|\Delta\tau|$, respectively (equivalent parameter values are shown in brackets). The case of all parameters being equal is labeled as "ep". The Fox theory predicts the same result (ep) in all cases. We consider a system **(a)** without and **(b)** with thermal noise; note the different scale on the vertical axes.

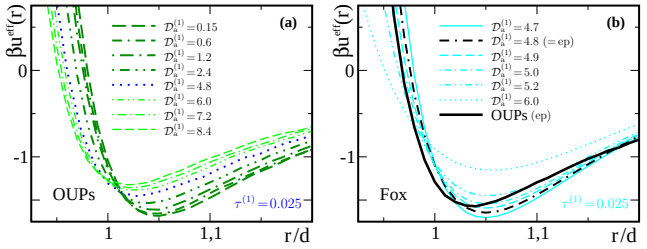


FIG. 2. As Fig. 1, but for two particles with fixed $\tau^{(1)} = 0.025$ and thus $\tau^{(2)} = 0.075$. We qualitatively compare **(a)** simulations to **(b)** theory, where $\Delta\mathcal{D}_{\rm a}$ is chosen much smaller for the theoretical curves, since the changes for different $\mathcal{D}_{\rm a}^{(1)}$ are more significant and the curves start to diverge for some $\mathcal{D}_{\rm a}^{(1)} < 4.8$. This is a direct consequence of the form of Eq. (42). The simulation result included in the theory plot and the line for $\mathcal{D}_{\rm a}^{(1)} = 4.8$ are the same as in Fig. 1 (left).

eigenvalues of the more general (inverse) diffusion tensor $\mathcal{D}_{ij}^{-1}$ from Eq. (13).

In the most general case of two different species with thermal noise present, we can express $u_{\upsilon\mu}^{\rm eff}$ from Eq. (37) with help of the eigenvalues

$$E_n^{(\upsilon\mu)}(r) = \frac{\bar{E}_n(r)}{\bar{\mathcal{D}}_{\rm a} - \Delta\mathcal{D}_{\rm a}\,\Delta E_n(r) + \bar{I}_{\rm t}\bar{E}_n(r)} \tag{40}$$

of $\mathcal{D}_{\rm m}^{-1}$ from Eq. (30) with Eq. (13), with the average and deviatoric parameters

$$\bar{\Xi} := \frac{\Xi^{(\upsilon)} + \Xi^{(\mu)}}{2}\,, \quad \Delta\Xi := \frac{\Xi^{(\upsilon)} - \Xi^{(\mu)}}{2}\,, \tag{41}$$

where $\Xi^{(\upsilon)}$ can stand for the eigenvalues $E_n^{(\upsilon)}$ from Eq. (39), the persistence times $\tau^{(\upsilon)}$, the active diffusivities $\mathcal{D}_{\rm a}^{(\upsilon)}$ or the characteristic functions $I_{\rm t}^{(\upsilon)}$ of thermal noise. It is a known problem of the present theory that, already for the one-component system, the effective potential is only well defined if the eigenvalues of the diffusion tensor are strictly positive. Systems for which this validity criterion is fulfilled, as, for example, soft-repulsive particles in one dimension, are rather the exception then the rule.

Regarding mixtures, we should expect that additional difficulties arise if the term $\Delta\mathcal{D}_{\rm a}\,\Delta E_n(r)$ in Eq. (40), which is not present in the result based on the UCNA, is positive. The nature of this term can be understood by the following example. In the present model we can define a passive (Brownian) species $\mu$ in two ways [38]: we always require that $E_n^{(\mu)} = 1$, i.e., $\tau^{(\mu)} = 0$ (likewise, with a finite persistence time, the bare potential is not recovered in the one-component UCNA [38]). Obviously, the persistence time $\tau^{(\upsilon)}$ of the active species has to remain finite. Then we can set either $\mathcal{D}_{\rm a}^{(\mu)} = 1$ with $I_{\rm t}^{(\mu)} = 0$ or $\mathcal{D}_{\rm a}^{(\mu)} = 0$ with $I_{\rm t}^{(\mu)} = 1$. Both definitions result in the same $E_n^{(\upsilon\mu)}$ for an active-passive mixture with arbitrary $\tau^{(\upsilon)}$ and $\mathcal{D}_{\rm a}^{(\upsilon)}$ only if the term $\Delta\mathcal{D}_{\rm a}\,\Delta E_n(r)$ is present.

In the following, we restrict ourselves to the one-dimensional case, where the one-component theory is highly accurate [38, 39]. Solving Eq. (37) we obtain the most general form

$$\beta u_{\upsilon\mu}^{\rm eff}(x) = \int_{-\infty}^{x} {\rm d}s\, \frac{d\beta u(s)}{ds}\, E_2^{(\upsilon\mu)}(s) - \ln\!\left(\bar{\mathcal{D}}_{\rm a}\,\left|E_2^{(\upsilon\mu)}(x)\right|\right) \tag{42}$$

of the effective potential for a symmetric bare potential with $\lim_{x\to\infty} u(x) = 0$. For two different species, it is not possible to carry out the integral in general, even if $I_{\rm t}^{(\upsilon)} = I_{\rm t}^{(\mu)} = 0$. If, in addition, both species either have the same active diffusivity $\mathcal{D}_{\rm a}^{(\upsilon)} = \mathcal{D}_{\rm a}^{(\mu)}$ or persistence time $\tau^{(\upsilon)} = \tau^{(\mu)}$, one recovers an effective potential similar to the single-component result in Eq. (38) with $\mathfrak{d} = 1$, replacing $\tau^{(\upsilon)} \to \bar{\tau}$ or $\mathcal{D}_{\rm a}^{(\upsilon)} \to \bar{\mathcal{D}}_{\rm a}$, respectively, with the appropriate average parameter.

It is straight-forward to generalize the above discussion to higher spatial dimensions. However, we would face the following complications: (i) the exact effective potentials can be written in a form similar to Eq. (42) only if [38], in addition to $I_{\rm t}^{(\upsilon)} = I_{\rm t}^{(\mu)} = 0$, we assume $\mathcal{D}_{\rm a}^{(\upsilon)} = \mathcal{D}_{\rm a}^{(\mu)}$ or $\tau^{(\upsilon)} = \tau^{(\mu)}$; (ii) we have no rigorous proof that the underlying assumption $\mathbf{J}_1 = \mathbf{J}_2$ holds for $\mathfrak{d} > 1$; (iii) already for two particles of the same species, empirical corrections of the effective potential are required and the quantitative agreement with computer simulation becomes worse with increasing dimensionality [38].

### D. Model calculations in one dimension

To test the generalized theory we consider a soft-repulsive bare potential $u(x) = x^{-12}$ between two particles and perform computer simulations of active

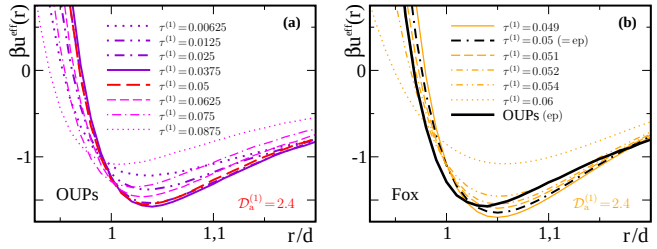


FIG. 3. As Fig. 1, but for two particles with fixed $\mathcal{D}_{\rm a}^{(1)}=2.4$ and thus $\mathcal{D}_{\rm a}^{(2)}=7.2$. As in Fig. 2, we compare **(a)** simulations to **(b)** theory, where the curves diverge for some $\tau^{(1)}<0.05$.

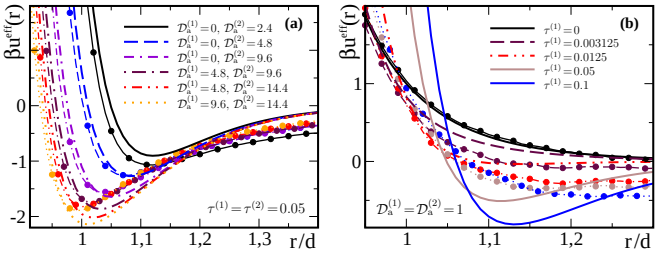


FIG. 4. Comparison between the effective potentials from theory (thick lines) and simulations (dots and thin lines) for **(a)** a fixed persistence time $\tau^{(1)}=\tau^{(2)}=0.05$ but different active diffusivities (as labeled) and **(b)** an active-passive mixture with fixed $\mathcal{D}_{\rm a}^{(1)}=\mathcal{D}_{\rm a}^{(2)}=1$ and the passive $\tau^{(1)}=0$ but different active persistence times (as labeled). In this case, the equilibration appears to proceed very slowly and the numerical values only gradually approach 0 for separations larger than shown here. However, even for large values of $\tau^{(2)}$ there is no attractive well, suggesting that the fully equilibrated data should reflect the behavior of two passive particles.

OUPs [38], evolving according to Eq. (1), as a benchmark for the effective potentials predicted from Fox's approach. Since we focus on the one-dimensional case, we can also make quantitative statements about whether the accuracy of the single-component theory [38, 47] is maintained if the difference in activity increases. We checked that the qualitative behavior is not altered when an additional thermal-noise vector is added to Eq. (1) and a quantitative comparison to the simulation data is thus analogous to the single-component case [38]. An exemplary direct comparison of these two systems is made in Fig. 1.

For the following discussions, recall the definitions and sign of relative parameters from Eq. (41), where the particle labeled 1 is always considered a member of species $v$. The most important theoretical statement of Sec. IV C is that either for equal $\mathcal{D}_{\rm a}^{(1)}=\mathcal{D}_{\rm a}^{(2)}=\bar{\mathcal{D}}_{\rm a}$ or $\tau^{(1)}=\tau^{(2)}=\bar{\tau}$ the effective potential of the mixture is equal to that of two identical particles with averaged activity parameters. According to Fig. 1 this prediction is indeed confirmed numerically for the latter case, whereas at equal $\mathcal{D}_{\rm a}$ the effective potential should rather become less attractive with increasing difference $|\Delta\tau|$. We will return to this point at the end of this section.

If the two species differ in both activity parameters, the theoretical results do no longer depend only on the average values due to the term $\Delta\mathcal{D}_{\rm a}\,\Delta E_2(r)$ in Eq. (42). For small differences $\Delta\mathcal{D}_{\rm a}$ and $\Delta\tau$, we observe in Figs. 2 and 3 that both theory and simulations predict a deeper minimum of the effective potential when $\Delta\mathcal{D}_{\rm a}\Delta\tau$ is increased. Further increasing the absolute value of either difference, the theory becomes quantitatively inaccurate. For $\Delta\mathcal{D}_{\rm a}\Delta\tau<0$ the theoretical curves suggest a rapid decline in the effective attraction, while much larger differences between the parameters are required to noticeably shift the numerical curves. For $\Delta\mathcal{D}_{\rm a}\Delta\tau>0$ the theory starts to predict diverging effective potentials, which is qualitatively wrong.

Interestingly, the behavior of the simulation results in the regime $\Delta\mathcal{D}_{\rm a}\Delta\tau>0$ distinctly depends on the parameters that are changed. Fixing $\Delta\tau<0$ and decreasing $\Delta\mathcal{D}_{\rm a}<0$ the curves in Fig. 2a begin to saturate and the deepest minimum is reached for the minimal $\Delta\mathcal{D}_{\rm a}$ i.e., $\mathcal{D}_{\rm a}^{(1)}=0$. On the other hand, the curve with the deepest minimum in Fig. 3a at constant $\Delta\mathcal{D}_{\rm a}<0$ is found for an intermediate $\Delta\tau<0$, whereas for even smaller $\Delta\tau$ the trend inverts. This means that for large absolute differences $|\Delta\tau|$ in the persistence time, i.e., one species becoming more and more passive, there is no significant attraction between two active OUPs (see also Fig. 1).

The special case of a common persistence time $\tau=\tau^{(1)}=\tau^{(2)}$ of both particles is particularly relevant, since $\tau$ represents a common rotational diffusion of all species in a mixture of ABPs with different self-propulsion velocities [46]. As noted before, this choice of parameters also yields particularly simple effective potentials, which are equivalent to the single-component results with the averaged diffusivity $\bar{\mathcal{D}}_{\rm a}$ defined according to Eq. (41). Recall from Fig. 1 that also computer simulation results are well represented by those with $\bar{\mathcal{D}}_{\rm a}$. Figure 4a nicely confirms for different activity parameters our expectation that for the Brownian mixture under consideration the effective potentials are as accurate as those between identical particles in one spatial dimension [38, 39]. Accordingly, the deviations from the simulation results are most significant for large separations $r/d$ and increase with increasing average activity $\bar{\mathcal{D}}_{\rm a}$.

Another special case, which recently has attracted much interest, is a mixture of an active and a passive Brownian species [24–26]. Given the prior results, the only way to set up a meaningful theoretical description of such a system is to fix $\mathcal{D}_{\rm a}^{(1)}=\mathcal{D}_{\rm a}^{(2)}=1$, since the persistence times $\tau^{(2)}=0$ of the passive and $\tau^{(1)}$ of the active species are different by necessity in the OUPs model. Choosing the latter as the free activity parameter, as shown in Fig. 4b, makes it difficult to connect to ABPs, where the activity should rather be tuned by varying the active diffusivity $\mathcal{D}_{\rm a}^{(1)}$ (depending on the self-propulsion velocity) than the (non-Brownian) reorienta-

tion time $\tau^{(1)}$.

In general, our OUPs simulations indicate that there is no significant effective attraction whenever one of the two particles is passive. Putting aside the difficulties with the equilibration, the numerical curves in Fig. 4b are practically independent of the activity of the second particle. Inspired by Percus' test particle approach [50], we conjecture that the pair distribution in a two-body system including a passive Brownian particle always reflects the behavior of the bare interaction potential, regardless of the type and magnitude of self propulsion of the other species. This conclusion is also consistent with the behavior of the different curves shown in Figs. 1 and 3a upon increasing $|\Delta\tau|$. As for these sets of parameters, the theoretical effective potentials in Fig. 4b overestimate the effective attraction. However, in this special case with one passive particle the simulation result obtained on the two-body level obviously does not reflect the behavior of the many-body system.

## V. CONCLUSIONS

In this paper, we derived the multicomponent generalization of the multidimensional Fox approximation and applied it to pairs of active particles with different persistence times and diffusivities. Our analytic results were compared to an exact solution for harmonic potentials and computer simulations for soft-repulsive interactions in one dimension, which both suggest that our theory is satisfactory at linear order in the persistence times. Explicitly, the formulas (42) with (40) for the effective interaction potentials between two different particles resemble those for two identical particles with average activity parameters plus an additional term depending on their differences. This term is important to make correct qualitative predictions close to equilibrium (or for small differences between the parameters) but also can be identified as the reason for the wrong or even unphysical predictions beyond the low-activity limit. It might be interesting in future work to explore the possibility for an empirical modification of the effective potential in the spirit of the inverse-$\tau$ approximation introduced in Ref. 38.

As for a single component, the predicted effective potentials can be readily implemented in a density functional theory in order to make explicit predictions of the structure and phase behavior [11]. Keeping this in mind, we also stress that the limitations discussed in Sec. IV D if the two particles belong to different species, will eventually only play a minor role on the many-particle level. Here, the effective pair potential between members of the same species will be equally or, most likely, even more relevant. The same is true if one is interested in a mixture in the presence of external forces, which are also single-body quantities. It might thus also constitute a fair approximation to drop the term $\Delta\mathcal{D}_\mathrm{a}\,\Delta E_2(r)$ in Eq. (40), i.e., to simply use the average parameters in any case.

The most obvious application of our effective potentials is also among those of most recent interest. Following the intuition from passive mixtures, the tendency of a two-component system to demix arises from differences in the interactions between the members of each species. In our case the increased effective attraction within the (more) active species can be interpreted as the driving force of the demixing process when its activity is increased. It will thus come to no surprise that an explicit (passive) calculation will predict that the effective equilibrium state of a mixture of two different active species is demixed/phase separated. Of course the problem of demixing is ill-defined in one dimension, but the results discussed here will qualitatively be the same in higher dimensions, although it is a well-known problem of the theory that the effective attraction is overestimated and further empirical corrections are necessary in order to avoid divergences [38].

Considering the problem of active depletion it is not as simple to draw conclusion solely from discussing the effective potentials. The expected attraction between the passive particles does not result directly from the (still passive) effective potential between members of this species. It is still quite likely that the enhancement of the depletion interaction is also captured in our theory and can be implicitly accounted for when studying the full behavior of an effective mixture between an active ideal gas higher effective temperature) and passive colloids. To show this explicitly, another calculation in the spirit of the Asakura-Oosawa model [51] would be necessary. Likewise, all other combinations of different passive potentials in the active mixture can be modeled within our approach, where the effective interactions between members of different species, as studied here, are generally important. As for a single component, an extension of the theory to study dynamical problems is straightforward.

## ACKNOWLEDGEMENTS

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### Appendix A: The multicomponent Fox approximation

We derive of the Fokker-Planck evolution equation of the probability density distribution associated with the stochastic differential equation (1) by employing a generalized Fox approximation [45, 46]. By differentiating with respect to time the probability density distribution (5) associated the stochastic process Eq. (1) we obtain

the following equation:

$$\frac{\partial f_N(\{y_\alpha\},t)}{\partial t} = -\sum_\beta \frac{\partial}{\partial y_\beta}\Bigg( D_{\rm t}\beta F_\beta(\{y_\alpha\}) f_N(\{y_\alpha\},t) + \int D[\{\chi_\alpha\}]P_N[\{\chi_\alpha\}]\Big(\prod_\alpha \delta(y_\alpha - x_\alpha(t))\Big)\chi_\beta(t)\Bigg). \tag{A1}$$

The first term in Eq. (A1) stems from the deterministic part of the evolution, whereas the second term accounts for the noise contribution and is calculated as follows. We first use the Novikov theorem and the explicit form of the noise correlation (3) and rewrite Eq. (A1) as:

$$\int D[\{\chi_\alpha\}]P_N[\{\chi_\alpha\}]\left(\prod_\alpha \delta(y_\alpha - x_\alpha(t))\right)\chi_\beta(t) = -\sum_\gamma \int {\rm d}t' C_{\beta\beta}(t-t')\frac{\partial}{\partial y_\gamma}\int D[\{\chi_\alpha\}]P_N[\{\chi_\alpha\}] \times\left(\prod_\alpha \delta(y_\alpha - x_\alpha(t))\right)\frac{\delta x_\gamma(t)}{\delta\chi_\beta(t')}. \tag{A2}$$

In order to evaluate the response function (the last factor) in the right-hand side of the above expression, we use again Eq. (1) and find

$$\frac{\delta \dot{x}_\gamma(t)}{\delta\chi_\beta(t')} = D_{\rm t}\sum_\delta \frac{\partial \beta F_\gamma(x^{\mathfrak{d}N}(t))}{\partial x_\delta(t)}\frac{\delta x_\delta(t)}{\delta\chi_\beta(t')} + \delta_{\beta\gamma}\delta(t-t'). \tag{A3}$$

The formal solution of Eq. (A3) with the initial condition

$$\left[\frac{\delta x_\gamma(t)}{\delta\chi_\beta(t')}\right]_{t=t'} = \delta_{\beta\gamma} \tag{A4}$$

is given (for $t > t'$) by the tensor

$$\frac{\delta x_\gamma(t)}{\delta\chi_\beta(t')} = \left(\exp\int_{t'}^t {\rm d}s\ \boldsymbol{\mathfrak{F}'}(s)\right)_{\gamma\beta}\Theta(t-t') \approx \Big(\exp(t-t')\ \boldsymbol{\mathfrak{F}'}(t)\Big)_{\gamma\beta}\Theta(t-t'). \tag{A5}$$

In the second step, we expanded the integral [46] in the exponent up to linear order in $(t-t')$ and we introduced $\boldsymbol{\mathfrak{F}'}(t) \simeq \boldsymbol{\mathfrak{F}'}[\{x_\alpha(t)\}]$ with the components $\boldsymbol{\mathfrak{F}'}_{\gamma\beta} = D_{\rm t}\partial\beta F_\gamma/\partial x_\beta$. To shorten the notation we indicate the average of a function $\mathcal{O}(x_\alpha(t))$ as

$$\langle \mathcal{O}(x_\alpha(t))\rangle \equiv \int D[\{\chi_\alpha\}]P_N[\{\chi_\alpha\}]\mathcal{O}(x_\alpha(t)) \tag{A6}$$

Now we use Eq. (A5) to rewrite Eq. (A2) as

$$-\frac{\partial}{\partial y_\gamma}\sum_\gamma \int_0^t {\rm d}t' C_{\beta\beta}(t-t')\Big\langle\prod_\alpha \delta(y_\alpha - x_\alpha(t))\frac{\delta x_\gamma(t)}{\delta\chi_\beta(t')}\Big\rangle \approx -\sum_\gamma \frac{\partial}{\partial y_\gamma} f_N(\{y_\alpha\},t)\int_0^t {\rm d}t' C_{\beta\beta}(t-t') \times\Big\langle\Big(e^{(t-t')\,\boldsymbol{\mathfrak{F}'}(t)}\Big)_{\gamma\beta}\Big\rangle, \tag{A7}$$

where $f_N \equiv \Big\langle \prod_\alpha \delta(y_\alpha - x_\alpha(t))\Big\rangle$ and we approximated the average of the product in the in the first line by the product of the averages. Using the explicit correlator (3) and further approximating the average of the exponential with the exponential of the average the integral featuring in Eq. (A7) becomes:

$$\int_0^t {\rm d}t' \frac{D_{\rm a}^{(\beta)}}{\tau_{\rm a}^{(\beta)}} e^{-\frac{|t-t'|}{\tau_{\rm a}^{(\beta)}}} \times\left(e^{(t-t')\left\langle\boldsymbol{\mathfrak{F}'}(t)\right\rangle}\right)_{\gamma\beta} \approx D_{\rm a}^{(\beta)}\Big(\mathbf{I} - \tau_{\rm a}^{(\beta)}\Big\langle\boldsymbol{\mathfrak{F}'}(t)\Big\rangle\Big)^{-1}_{\gamma\beta} \tag{A8}$$

where we took the small $\tau_{\rm a}^{(\beta)}$ limit in the integral. Putting together

$$\frac{\partial f_N(\{y_\alpha\},t)}{\partial t} = -\sum_\beta \frac{\partial}{\partial y_\beta}\Bigg( D_{\rm t}\beta F_\beta(\{y_\alpha\}) f_N(\{y_\alpha\},t) - D_{\rm a}^{(\beta)}\sum_\gamma \frac{\partial}{\partial y_\gamma} f_N(\{y_\alpha\},t)\Big(\mathbf{I} - \tau_{\rm a}^{(\beta)}\Big\langle\boldsymbol{\mathfrak{F}'}(t)\Big\rangle\Big)^{-1}_{\gamma\beta}\Bigg) \tag{A9}$$

Such a formula can be recast under the form (6) given in the main text.


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