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A Simple New Method for Calculating the Characters of the Symmetric Groups

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ABSTRACT

The method of Young diagrams for the symmetric groups is reformulated with special emphasis on the interrelationships among all the different primitive idempotents generated by the Young tableaux. A fundamental connection is found between the idempotent of an irreducible representation and all the primitive idempotents generated by the different Young tableaux associated with the irreducible representation. This result and its associated theorems are used to solve the problem of obtaining the irreducible characters from the Frobenius compound character formula. The final procedure is surprisingly simple.

1. INTRODUCTION

Several of the results given in Section II are not new and come directly from Young's original method. However, they are considerably made use of in the subsequent sections and are therefore included without proof. Whenever a result was thought to be new by the author, its proof was included. The important theorems the theory is able to produce are Theorems 3.1 and 5.1.

2. THE SYMMETRIC GROUPS AND YOUNG TABLEAUX

In any finite group conjugate elements have the same cycle structure.[1]

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[1] For arbitrary finite groups consider the action of an element of the group on the

In the symmetric groups elements with the same cycle structure are conjugate. Therefore, in the symmetric groups the classes of elements with the same cycle structure and the conjugate classes are the same. Using Greek letters to denote cycle structures, the cycle structure α can be given explicitly by

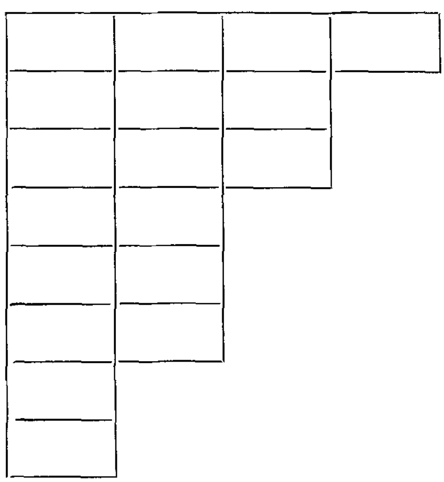
$$\alpha = (l_1^{m_1}, l_2^{m_2}, l_3^{m_3}, ..., l_p^{m_p}),$$

where $l_1 > l_2 \cdots > l_p$ and the l_i are the cycle lengths while the m_i are their multiplicities, or by

$$\alpha = (\lambda_1, \lambda_2, \lambda_3, ..., \lambda_q),$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_q$ and there are $m_i\, \lambda_i$ of length l_i for each value of i. The conjugate classes are to be indexed by the Greek letter, which corresponds to the cycle structure that determines the conjugate class. Thus, every element in conjugate class C_α has cycle structure α.

A frame is a two-dimensional rectangular array of boxes used to represent a given cycle structure.[2] The frame representing the cycle structure α has λ_1 boxes in its first row, λ_2 boxes in its second row, and generally λ_j boxes in its j-th row where $\alpha = (\lambda_1, \lambda_2, ..., \lambda_j, ..., \lambda_q)$. As an example, consider the symmetric group of order 18, S_{18}. The frame representing the cycle structure $\alpha = (4, 3, 3, 2, 2, 2, 1, 1) \equiv (4^1, 3^2, 2^3, 1^2)$ is:



group as a permutation of the elements of the group. The cycle structure of the permutation determined is the cycle structure referred to above.

[2] The use of frames and tableaux for the symmetric groups was initiated by Alfred

Given a cycle structure, α, and its frame, the columns of the frame determine another cycle structure, β, which is called the dual of α and is denoted by $\bar{\alpha}$. If

$$\alpha = (l_1^{m_1}, l_2^{m_2}, \ldots, l_p^{m_p}),$$

then

$$\bar{\alpha} = (\bar{l}_1^{\bar{m}_1}, \bar{l}_2^{\bar{m}_2} \ldots, \bar{l}_p^{\bar{m}_p}),$$

where $p = q$ and

$$\bar{l}_i = \sum_{j=1}^{p+1-i} m_j$$

and

$$\bar{m}_j = l_{p+1-i} - l_{p+2-i},$$

with $l_{p+1} = 0$. In the example from S_{18} given above it follows that $\bar{\alpha} = (8^1, 6^1, 3^1, 1^1)$. This can be verified with the above identities or directly from the frame for α.

If a frame has a total of m boxes, a tableaux is constructible from the frame by placing the first m natural numbers in the boxes, one integer per box. Since this can be done in $m!$ ways, there are $m!$ tableaux constructible from each frame. The canonical tableaux is the one constructed by placing the numbers in increasing order as one goes from left to right along the first row and then from left to right along the second row and so on. For the example of $\alpha = (4^1, 3^2, 2^3, 1^2)$ in S_{18} the canonical tableaux is

1	2	3	4
5	6	7	
8	9	10	
11	12		
13	14		
15	16		
17			
18			

Young in his brilliant work on the theory of invariants during the early part of this century.

For S_N the canonical tableaux constructed from the frame associated with the cycle structure α is denoted by T^α. For $g \in S_N$ the operation of g on T^α is defined by

$$gT^\alpha = T_g{}^\alpha,$$

where $T_g{}^\alpha$ is the tableaux obtained from T^α by performing the permutation on the entries of T^α which is dictated by g. In the case of the example of $\alpha = (4^1, 3^2, 2^3, 1^2)$ in S_{18}, T^α has already been given. Let

$$g \in S_{18} \text{ be } g = (1, 9, 5, 17)\,(14, 13, 6)\,(15, 16, 8, 11, 18)\,(2, 10)\,(4, 12, 3, 7).$$

Then it follows that $gT^\alpha = T_g{}^\alpha$ is given by

17	10	12	7
9	13	3	
16	1	2	
8	4		
14	6		
18	15		
5			
11			

It is trivial that

$$T^\alpha = eT^\alpha = T_e{}^\alpha,$$

where e is the identity element of S_N. In general the tableaux satisfy

$$T_g{}^\alpha = gh^{-1}T_h{}^\alpha.$$

Since the order of S_N is $N!$ and there are $N!$ tableaux which can be constructed for each frame associated with a cycle structure of S_N, and for g, $h \in S_N$ with $g \neq h$, it follows that $T_g{}^\alpha \neq T_h{}^\alpha$; then the set of all possible distinct tableaux constructible from a given frame is $\{gT^\alpha \mid g \in S_N\}$. The set $\{gT^\alpha \mid g \in S_N\}$ is also identically $\{T_g{}^\alpha \mid g \in S_N\}$.

In the following, restrict attention to S_N of arbitrary but fixed order $N!$ and to an arbitrary but fixed cycle structure α in S_N. Associated with α is a frame. Constructible from this frame is the set of tableaux, $\{T_g{}^\alpha \mid g \in S_N\}$.

The row content, $R_g{}^i$, of the i-th row of $T_g{}^\alpha$ is defined to be the set of integers in the i-th row, independent of their order in the row. The content of the rows of $T_g{}^\alpha$, $C_r(T_g{}^\alpha)$, is defined to be the ordered set of row contents of $T_g{}^\alpha$. That is

$$C_r(T_g{}^\alpha) = \{R_g{}^1, R_g{}^2, ..., R_g{}^p\},$$

wherein the order is respected. Similarly, the content of the columns of $T_g{}^\alpha$, $C_c(T_g{}^\alpha)$, is defined to be the ordered set of column contents, $C_g{}^i$, of $T_g{}^\alpha$. That is

$$C_r(T_g{}^\alpha) = \{C_g{}^1, C_g{}^2, ..., C_g{}^q\}.$$

Define $H_g{}^\alpha$ by

$$H_g{}^\alpha = \{s \in S_N \mid C_r(sT_g{}^\alpha) = C_r(T_{sg}^\alpha) = C_r(T_g{}^\alpha)\}$$

and $V_g{}^\alpha$ by

$$V_g{}^\alpha = \{s \in S_N \mid C_c(sT_g{}^\alpha) = C_c(T_{sg}^\alpha) = C_c(T_g{}^\alpha)\}.$$

$H_g{}^\alpha$ is called the horizontal group of $T_g{}^\alpha$ and $V_g{}^\alpha$ is called the vertical group of $T_g{}^\alpha$. Define $S_g{}^\alpha$ by

$$S_g{}^\alpha = \sum_{s \in H^\alpha{}_g} s$$

and $A_g{}^\alpha$ by

$$A_g{}^\alpha = \sum_{s \in V^\alpha{}_g} \varepsilon_s s,$$

where $\varepsilon_s = +1$ for even s and $\varepsilon_s = -1$ for odd s. $S_g{}^\alpha$ is called the Young symmetrizer of $T_g{}^\alpha$ and $A_g{}^\alpha$ is called the Young anti-symmetrizer of $T_g{}^\alpha$. Define $E_g{}^\alpha$ by

$$E_g{}^\alpha = S_g{}^\alpha A_g{}^\alpha.$$

THEOREM 2.1.[3] *Given* $T_g{}^\alpha$, $H_g{}^\alpha$, $V_g{}^\alpha$, $S_g{}^\alpha$, $A_g{}^\alpha$, $E_g{}^\alpha$ and $T_h{}^\alpha$, $H_h{}^\alpha$, $V_h{}^\alpha$, $S_h{}^\alpha$, $A_h{}^\alpha$, $E_h{}^\alpha$, *then*

$$T_g{}^\alpha = gh^{-1}T_h{}^\alpha$$

[3] Theorems 2.1, 2.2, and 2.3 are offered without proof since they merely summarize some of Young's work.

$$H_g^\alpha = (gh^{-1})H_h^\alpha(gh^{-1})^{-1}$$
$$V_g^\alpha = (gh^{-1})V_h^\alpha(gh^{-1})^{-1}$$
$$S_g^\alpha = (gh^{-1})S_h^\alpha(gh^{-1})^{-1}$$
$$A_g^\alpha = (gh^{-1})A_h^\alpha(gh^{-1})^{-1}$$
$$E_g^\alpha = (gh^{-1})E_h^\alpha(gh^{-1})^{-1}$$

THEOREM 2.2.[3] *E_g^α is essentially idempotent; that is $(E_g^\alpha)^2 = \lambda_g^\alpha E_g^\alpha$ where λ_g^α is some real number.*

COROLLARY. *$(1/\lambda_g^\alpha)\, E_g^\alpha$ is a primitive idempotent of the ring $R(S_N)$.*

THEOREM 2.3.[3] *In the ring $R(S_N)$, the $N!$ minimal ideals generated by the $N!$ primitive idempotents, $(1/\lambda_g^\alpha)\, E_g^\alpha$ for $g \in S_N$, span a simple two-sided ideal.*

LEMMA 2.1. *The λ_g^α determined by $(E_g^\alpha)^2 = \lambda_g^\alpha E_g^\alpha$ are independent of the subscript g.*

PROOF: Let $g, h \in S_N$ with $g \neq h$. By Theorem 2.1

$$E_g^\alpha = (gh^{-1})E_h^\alpha(gh^{-1})^{-1}.$$

Therefore,

$$\begin{aligned}\lambda_g^\alpha E_g^\alpha &= (E_g^\alpha)^2\\ &= [(gh^{-1})E_h^\alpha(gh^{-1})^{-1}]^2\\ &= (gh^{-1})(E_h^\alpha)^2(gh^{-1})^{-1}\\ &= \lambda_h^\alpha(gh^{-1})E_h^\alpha(gh^{-1})^{-1}\\ &= \lambda_h^\alpha E_g^\alpha\end{aligned}$$

Therefore, $\lambda_g^\alpha = \lambda_h^\alpha$, which proves the lemma.

Since λ_g^α is independent of g, it can be denoted more simply by λ^α. Theorem 2.2 and Lemma 2.1 show that for all $g \in S_N$, $(1/\lambda^\alpha)\, E_g^\alpha$ is a primitive idempotent of $R(S_N)$. In the following $(1/\lambda^\alpha)\, E_g^\alpha$ will be denoted by e_g^α.

3. RELATIONSHIPS AMONG THE IDEMPOTENTS

It is a simple fact that σ is a primitive idempotent in $R(S_N)$, iff for every $r \in R(S_N)$

$$\sigma\, r\, \sigma = \xi_r^\sigma\, \sigma$$

where ξ_r^σ is a real number.

LEMMA 3.1. *Given T^a and $g, h \in S_N$ such that $g \neq h$, then $e_g{}^a e_h{}^a$ is one of*: (1) *essentially a primitive idempotent*, (2) *nilpotent*,[4] *or* (3) *null.*

PROOF: Let e^a be the primitive idempotent constructible from the canonical tableaux, T^a. By Theorem 2.1 it follows that

$$e_g{}^a = g e^a g^{-1} \qquad \text{and} \qquad e_h{}^a = h e^a h^{-1}.$$

By virtue of the observation which began this section,

$$e_g{}^a e_h{}^a = g e^a g^{-1} h e^a h^{-1} = \xi_{g^{-1}h}\, g e^a h^{-1}.$$

Therefore

$$\begin{aligned}(e_g{}^a e_h{}^a)^2 &= (\xi_{g^{-1}h})^2 g e^a h^{-1} g e^a h^{-1} \\ &= (\xi_{g^{-1}h})^2 \xi_{h^{-1}g}\, g e^a h^{-1} \\ &= \xi_{g^{-1}h} \xi_{h^{-1}g} (e_g{}^a e_h{}^a).\end{aligned}$$

If $\xi_{g^{-1}h} = 0$ then (3) holds. If $\xi_{h^{-1}g} = 0$ then (2) holds. If $\xi_{g^{-1}h}\xi_{h^{-1}g} \neq 0$ then $e_g{}^a e_h{}^a$ is essentially idempotent. For any $r \in R(S_N)$

$$\begin{aligned}(e_g{}^a e_h{}^a) r (e_g{}^a e_h{}^a) &= (\xi_{g^{-1}h})^2 g e^a h^{-1} r g e^a h^{-1} \\ &= (\xi_{g^{-1}h})^2 \xi_{h^{-1}rg}\, g e^a h^{-1} \\ &= \xi_{g^{-1}h} \xi_{h^{-1}rg} (e_g{}^a e_h{}^a).\end{aligned}$$

Therefore, $e_g{}^a e_h{}^a$ is primitive. Thus, for $\xi_{g^{-1}h}\xi_{h^{-1}g} \neq 0$, (1) holds.

LEMMA 3.2. *Given T^a and $g, h \in S_N$ such that $g \neq h$, either $H_g{}^a \cap V_h{}^a = \{e\}$ where e is the identity element of S_N or $H_g{}^a \cap V_h{}^a$ contains a transposition.*

PROOF. Either $H_g{}^a \cap V_h{}^a = \{e\}$ or $H_g{}^a \cap V_h{}^a$ contains some element $s \in S_N$. Write s as a product of transpositions. Assume that (i, j) is one of the transpositions in the product. Then from the definitions of $H_g{}^a$ and $V_h{}^a$ it follows that $(i, j) \in H_g{}^a$ and $(i, j) \in V_h{}^a$. This completes the proof.

LEMMA 3.3. *Given T^a and $g, h \in S_N$ such that $g \neq h$, if $H_g{}^a \cap V_h{}^a$ contains a transposition, t, then*

$$A_h{}^a S_g{}^a = 0.$$

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[4] Nilpotent, here and throughout, will mean that the square of the nilpotent quantity vanishes.

PROOF. The proof is omitted.[5]

COROLLARY. *If* $H_g^a \cap V_h^a \neq \{e\}$ *and* $H_h^a \cap V_g^a \neq \{e\}$ *then both* $e_g^a e_h^a$ *and* $e_h^a e_g^a$ *are null.*

PROOF. $H_g^a \cap V_h^a \neq \{e\}$ implies that there exists a transposition, $t \in H_g^a$ and $t \in V_h^a$, by Lemma 3.2. Therefore, by Lemma 3.3

$$\begin{aligned} e_h^a e_g^a &= (1/\lambda^a)^2 S_h^a A_h^a S_g^a A_g^a \\ &= 0. \end{aligned}$$

That $e_g^a e_h^a$ is also null follows from

$$H_h^a \cap V_g^a \neq \{e\}$$

in an identical manner.

LEMMA 3.4. *If* $H_g^a \cap V_h^a \neq \{e\}$ *and* $H_h^a \cap V_g^a = \{e\}$, *then* $e_h^a e_g^a$ *is null and* $e_g^a e_h^a$ *is nilpotent.*

PROOF. That $e_h^a e_g^a$ is null follows from the proof to the corollary to Lemma 3.3,

$$\begin{aligned} (e_g^a e_h^a)^2 &= e_g^a e_h^a e_g^a e_h^a \\ &= e_g^a (e_h^a e_g^a) e_h^a = 0. \end{aligned}$$

Therefore, $e_g^a e_h^a$ is nilpotent.

LEMMA 3.5. *If* $H_g^a \cap V_h^a = \{e\}$, *then*

$$T_h^a = r T_g^a,$$

where $r = pq$ *with* $p \in H_g^a$ *and* $q \in V_g^a$.

PROOF. The proof is omitted.[5]

LEMMA 3.6. *If* $H_g^a \cap V_h^a = \{e\}$ *and* $H_h^a \cap V_g^a = \{e\}$, *then there exists* l, $k \in S_N$ *such that*

$$e_g^a e_h^a = \pm e_l^a \text{ and } e_h^a e_g^a = \pm e_k^a.$$

[5] The proof is straightforward and can be found in any treatment of Young's method. Also, the proof to Lemma 3.3 is like that to lemma 4.1.

PROOF. By Lemma 3.5

$$H_g^a \cap V_h^a = \{e\}$$

implies

$$T_h^a = pqT_g^a,$$

where $p \in H_g^a$ and $q \in V_g^a$. However, by Theorem 2.1

$$T_h^a = hg^{-1}T_g^a,$$

so that

$$hg^{-1} = pq.$$

Again using Lemma 3.5

$$H_h^a \cap V_g^a = \{e\}$$

implies

$$T_g^a = \bar{p}\,\bar{q}\,T_h^a,$$

where $\bar{p} \in H_h^a$ and $\bar{q} \in V_h^a$. However, Theorem 2.1 gives

$$T_g^a = gh^{-1}T_h^a = (hg^{-1})^{-1}T_h^a$$

so that

$$(hg^{-1})^{-1} = \bar{p}\,\bar{q}.$$

Theorem 2.1 also states that H_g^a and V_g^a are given by

$$H_g^a = (hg^{-1})^{-1}H_h^a(hg^{-1})$$

and

$$V_g^a = (hg^{-1})^{-1}V_h^a(hg^{-1}).$$

Therefore,

$$p = (hg^{-1})^{-1}\bar{\bar{p}}(hg^{-1})$$

and

$$q = (hg^{-1})^{-1}\bar{\bar{q}}(hg^{-1})$$

for some $\bar{\bar{p}} \in H_h^a$ and some $\bar{\bar{q}} \in V_h^a$. Thus, $hg^{-1} = pq$ becomes

$$hg^{-1} = (hg^{-1})^{-1}\bar{\bar{p}}\bar{\bar{q}}(hg^{-1}),$$

which implies

$$hg^{-1} = \bar{\bar{p}}\,\bar{\bar{q}}.$$

The two equations, $(hg^{-1})^{-1} = \bar{p}\,\bar{q}$ and $hg^{-1} = \bar{\bar{p}}\,\bar{\bar{q}}$, imply

$$hg^{-1} = \bar{q}^{-1}\bar{p}^{-1} \qquad \text{and} \qquad (hg^{-1})^{-1} = \bar{\bar{q}}^{-1}\bar{\bar{p}}^{-1}.$$

Moreover, since $\bar{p}$ and $\bar{\bar{p}}$ are in $H_h{}^a$ and $\bar{q}$ and $\bar{\bar{q}}$ are in $V_h{}^a$, then

$$\bar{p}S_h{}^a = S_h{}^a = \bar{\bar{p}}S_h{}^a$$

and

$$A_h{}^a\bar{q} = \varepsilon_{\bar{q}}A_h{}^a \qquad \text{and} \qquad A_h{}^a\bar{\bar{q}} = \varepsilon_{\bar{\bar{q}}}A_h{}^a.$$

Therefore

$$\bar{p}e_h{}^a = e_h{}^a = \bar{\bar{p}}e_h{}^a$$

and

$$e_h{}^a\bar{q} = \varepsilon_{\bar{q}}e_h{}^a \qquad \text{and} \qquad e_h{}^a\bar{\bar{q}} = \varepsilon_{\bar{q}}e_h{}^a.$$

Using $\varepsilon_q = \varepsilon_q^{-1}$ and Theorem 2.1 again,

$$\begin{aligned} e_g{}^a e_h{}^a &= (hg^{-1})^{-1}e_h{}^a(hg^{-1})e_h{}^a \\ &= \bar{\bar{q}}^{-1}\bar{\bar{p}}^{-1}e_h{}^a\bar{q}^{-1}\bar{p}^{-1}e_h{}^a \\ &= \bar{\bar{q}}^{-1}e_h{}^a\varepsilon_{\bar{q}} \\ &= \varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}\bar{\bar{q}}^{-1}e_h{}^a\bar{\bar{q}} \\ &= \varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}e^a_{\bar{\bar{q}}^{-1}h}\,. \end{aligned}$$

Therefore, letting $l = \bar{\bar{q}}^{-1}h$,

$$e_g{}^a e_h{}^a = \pm\, e_l{}^a,$$

where the sign is given by $\varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}$.
Similarly,

$$\begin{aligned} e_h{}^a e_g{}^a &= e_h{}^a(hg^{-1})^{-1}e_h{}^a(hg^{-1}) \\ &= e_h{}^a\bar{\bar{q}}^{-1}\bar{\bar{p}}^{-1}e_h{}^a\bar{q}^{-1}\bar{p}^{-1} \\ &= \varepsilon_{\bar{\bar{q}}}\varepsilon_{\bar{q}}e_h{}^a\bar{p}^{-1} \\ &= \varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}\bar{p}e_h{}^a\bar{p}^{-1} \\ &= \varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}e^a_{\bar{p}h}\,. \end{aligned}$$

Therefore, letting $k = \bar{p}h$,

$$e_h{}^a e_g{}^a = \pm\, e_k{}^a,$$

where the sign is given by $\varepsilon_{\bar{q}}\varepsilon_{\bar{\bar{q}}}$. This completes the proof.

Lemmas 3.6, 3.4, and the corollary to Lemma 3.3 require that, for $e_g{}^a e_h{}^a$ to be nilpotent, it is necessary and sufficient that $H_g{}^a \cap V_h{}^a \neq \{e\}$ and $H_h{}^a \cap V_g{}^a = \{e\}$. This permits the following lemma.

LEMMA 3.7. *If $e_g{}^a e_h{}^a$ is nilpotent, then there exists $e_l{}^a$ and $e_k{}^a$ with $l \neq g$ and $k \neq h$ such that*

$$e_l{}^a e_k{}^a = \pm e_g{}^a e_h{}^a.$$

PROOF: The preceding remark requires that

$$H_g{}^a \cap V_h{}^a \neq \{e\}.$$

Let $s \in H_g{}^a \cap V_h{}^a$ with $s \neq e$. Therefore,

$$se_g{}^a = e_g{}^a \qquad \text{and} \qquad e_h{}^a s^{-1} = \varepsilon_s e_h{}^a,$$

which implies

$$\begin{aligned} e_g{}^a e_h{}^a &= \varepsilon_s s e_g{}^a e_h{}^a s^{-1} \\ &= \varepsilon_s s e_g{}^a s^{-1} s e_h{}^a s^{-1} \\ &= \varepsilon_s e^a_{sg} e^a_{sh} . \end{aligned}$$

Let $l = sg$ and $k = sh$. Since $s \neq e$, then $l \neq g$ and $k \neq h$ and

$$e_l{}^a e_k{}^a = \pm e_g{}^a e_h{}^a,$$

where the sign is given by ε_s.

LEMMA 3.8. *If $e_g{}^a e_h{}^a$ is nilpotent, then define $P^a_{g,h}$ and $N^a_{g,h}$ by*[6]

$$P^a_{g,h} = \{e_l{}^a e_k{}^a \quad \text{for } l, k \in S_N \mid e_l{}^a e_k{}^a = + e_g{}^a e_h{}^a\}$$

and

$$N^a_{g,h} = \{e_l{}^a e_k{}^a \quad \text{for } l, k \in S_N \mid e_l{}^a e_k{}^a = - e_g{}^a e_h{}^a\}.$$

If $\nu(X)$ denotes the number of elements in a set X, then

$$\nu(P^a_{g,h}) = \nu(N^a_{g,h}).$$

PROOF. Suppose $e_l{}^a e_k{}^a \in P^a_{g,h}$. By definition

$$e_l{}^a e_k{}^a = + e_g{}^a e_h{}^a.$$

[6] Strictly speaking, the sets described are meant to be those determined by

$$P^a_{gh} = \{(l, k) \in S_N X S_N \mid e_l{}^a e_k{}^a = + e_g{}^a e_h{}^a\} \text{ etc.}$$

With this in mind, it is convenient to write them as above.

Lemma 3.2 guarantees that $H_g{}^a \cap V_h{}^a$ contains a transposition t since $e_g{}^a e_h{}^a$ is nilpotent. The proof to Lemma 3.7 shows that

$$te_g{}^a e_h{}^a t^{-1} = - e_g{}^a e_h{}^a.$$

However, from above it follows that

$$\begin{aligned} te_g{}^a e_h{}^a t^{-1} &= te_l{}^a e_k{}^a t^{-1} \\ &= te_l{}^a t^{-1} te_k{}^a t^{-1} \\ &= e_{tl}^a e_{tk}^a . \end{aligned}$$

Therefore

$$e_{tl}^a e_{tk}^a = - e_g{}^a e_h{}^a,$$

which implies $e_{tl}^a e_{tk}^a \in N_{g,h}^a$. Since $tl = tl'$ and $tk = tk'$ iff $l = l'$ and $k = k'$, then conjugation by t as done above produces a one-to-one mapping of $P_{g,h}^a$ into $N_{g,h}^a$. Therefore

$$\nu(P_{g,h}^a) \leq \nu(N_{g,h}^a).$$

Since the same argument may be used to map $N_{g,h}^a$ into $P_{g,h}^a$ in a one-to-one way then

$$\nu(N_{g,h}^a) \leq \nu(P_{g,h}^a),$$

from which it is concluded that

$$\nu(P_{g,h}^a) = \nu(N_{g,h}^a).$$

LEMMA 3.9. *Define $X_g{}^a$ and $Y_g{}^a$ by*

$$X_g{}^a = \{e_l{}^a e_k{}^a \quad \text{for } l, k \,\varepsilon\, S_N \mid e_l{}^a e_k{}^a = + e_g{}^a\}$$

and

$$Y_g{}^a = \{e_l{}^a e_k{}^a \quad \text{for } l, k \in S_N \mid e_l{}^a e_k{}^a = - e_g{}^a\}.$$

For any pair $g, h \in S_N$ it follows that

$$\nu(X_g{}^a) = \nu(X_h{}^a)$$

and

$$\nu(Y_g{}^a) = \nu(Y_h{}^a).$$

PROOF: By Theorem 2.1

$$e_g{}^a = se_h{}^a s^{-1},$$

where $s = gh^{-1}$. Suppose $e_l{}^a e_k{}^a \in X_h{}^a$. By definition

$$e_l{}^a e_k{}^a = e_h{}^a.$$

Therefore

$$s e_l{}^a e_k{}^a s^{-1} = s e_h{}^a s^{-1}$$

or

$$\begin{aligned} e_g{}^a &= s e_l{}^a s^{-1} s e_k{}^a s^{-1} \\ &= e_{sl}^a \, e_{sk}^a . \end{aligned}$$

Therefore, conjugation by s produces a one-to-one mapping of $X_h{}^a$ into $X_g{}^a$, which implies

$$\nu(X_h{}^a) \leq \nu(X_g{}^a).$$

$X_g{}^a$ is mapped into $X_h{}^a$ by conjugation with $s = hg^{-1}$ giving

$$\nu(X_g{}^a) \leq \nu(X_h{}^a),$$

from which it follows that

$$\nu(X_g{}^a) = \nu(X_h{}^a).$$

In an entirely similar way it is shown also that

$$\nu(Y_g{}^a) = \nu(Y_h{}^a).$$

THEOREM 3.1. *$a^a = \sum_{g \in S_N} e_g{}^a$ is essentially idempotent and generates the simple two-sided ideal spanned by the $N!$ $e_g{}^a$'s.*

PROOF. The second half of the theorem follows directly from Theorem 2.3 provided the first half is true.

Consider

$$(a^a)^2 = \sum_{g \in S_N} \sum_{h \in S_N} e_g{}^a e_h{}^a.$$

According to Lemmas 3.6, 3.4, and the corollary to Lemma 3.3, three kind of terms appear in the above sum. The null terms make no contribution to the sum. For fixed $r, s \in S_N$, Lemma 3.8 guarantees that the sum of all nilpotent terms equal to $\pm e_r{}^a e_s{}^a$ is zero. Therefore, the sum of all nilpotent terms in $(a^a)^2$ is zero. This leaves only idempotent terms. Therefore, $(a^a)^2$ may be written as

$$(a^a)^2 = \sum_{g \in S_N} \omega_g{}^a e_g{}^a$$

where the ω_g^a's are positive or negative integers. By Lemma 3.9 ω_g^a is seen to be given explicitly by

$$\omega_g^a = \nu(X_g^a) - \nu(Y_g^a).$$

However, Lemma 3.9 showed that this expression is independent of the subscript g. Therefore, for any pair g, $h \in S_N$

$$\omega_g^a = \omega_h^a = \omega^a.$$

This gives

$$(a^a)^2 = \omega^a \sum_{g \in S_N} e_g^a = \omega^a a^a,$$

which proves that a^a is essentially idempotent.

LEMMA 3.10. $\omega^a \neq 0$.

PROOF: $\omega^a \equiv \omega_g^a = \nu(X_g^a) - \nu(Y_g^a)$ for any $g \in S_N$. We shall show that X_g^a is always larger than Y_g^a. It was shown by Lemma 3.6 that if $e_l^a e_k^a = \lambda e_g^a$ where $\lambda = \pm 1$ then $e_k^a e_l^a = \lambda e_h^a$ for some h. Note that λ is either $+1$ in both cases or -1 in both cases. From $\sigma\, r\, \sigma = \xi_r^\sigma \sigma$ it follows that $e_l^a e_k^a e_l^a = \xi_k^l e_l^a$ and $e_k^a e_l^a e_k^a = \xi_l^k e_k^a$. From above,

$$e_g^a = (\lambda e_g^a)(\lambda e_g^a) = e_l^a e_k^a e_l^a e_k^a = \begin{cases} \xi_k^l e_l^a e_k^a = \xi_k^l \lambda e_g^a \\ \xi_l^k e_l^a e_k^a = \xi_l^k \lambda e_g^a \end{cases}.$$

Therefore, $\xi_k^l \lambda = 1 = \xi_l^k \lambda$, $\xi_k^l = \lambda = \xi_l^k$, $e_l^a e_k^a e_l^a = \lambda e_l^a$, and $e_k^a e_l^a e_k^a = \lambda e_k^a$. Note again that $\lambda = \pm 1$, but that, once λ is determined by $e_l^a e_k^a = \lambda e_g^a$, it is fixed for all other relations derived. Taking products of e_g^a and e_h^a gives

$$e_g^a e_h^a = (\lambda e_g^a)(\lambda e_h^a) = e_l^a e_k^a e_k^a e_l^a = e_l^a e_k^a e_l^a = \lambda e_l^a$$

and by a similar argument $e_h^a e_g^a = \lambda e_k^a$. In summary, $e_l^a e_k^a = \lambda e_g^a$ implies:

$$e_k^a e_l^a = \lambda e_h^a$$

$$e_g^a e_h^a = \lambda e_l^a$$

$$e_h^a e_g^a = \lambda e_k^a$$

$$\lambda e_g^a = e_l^a e_k^a = \lambda e_l^a e_h^a e_g^a = e_l^a e_k^a e_l^a e_g^a = \lambda e_l^a e_g^a.$$

Therefore, $e_l^\alpha e_g^\alpha = e_g^\alpha$. Similarly, $e_g e_k^\alpha = e_g^\alpha$. The power of this result is that for

$$
\begin{aligned}
&l = g = k, &\quad& e_l^\alpha e_k^\alpha = + e_g^\alpha \\
&l = g \neq k, && e_l^\alpha e_k^\alpha = e_g^\alpha e_k^\alpha = + e_g^\alpha \\
&l \neq g = k, && e_l^\alpha e_k^\alpha = e_l^\alpha e_g^\alpha = + e_g^\alpha \\
&l \neq g \neq k, && e_l^\alpha e_g^\alpha = + e_g^\alpha \quad \text{and} \quad e_g^\alpha e_k^\alpha = + e_g^\alpha \ \{e_g^\alpha e_k^\alpha = + e_g^\alpha\}.
\end{aligned}
$$

This list shows that for $e_l^\alpha e_k^\alpha \in Y_g^\alpha$ it is necessary that $l \neq g \neq k$ in order to give $e_l^\alpha e_k^\alpha = - e_g^\alpha$. But, even if this is the case, associated with $e_l^\alpha e_k^\alpha$ are two members of X_g^α, namely, $e_l^\alpha e_g^\alpha = + e_g^\alpha$ and $e_g^\alpha e_k^\alpha = + e_g^\alpha$. Therefore, with each distinct member of Y_g^α we can associate two distinct members of X_g^α. That X_g^α is not empty follows from the $l = g = k$ case. Therefore, X_g^α is always larger than Y_g^α.

The idempotent $(1/\omega^\alpha)\, a^\alpha$ will be denoted by P^α.

The results found in the remaining part of this section will refer to idempotents generated by two distinct cycle structures α and β. These results make use of a linear order defined on the set of cycle structures in the following way. Let

$$\alpha = (\lambda_1^\alpha, \lambda_2^\alpha, \ldots, \lambda_p^\alpha)$$

where $\lambda_1^\alpha \geq \lambda_2^\alpha \geq \cdots \geq \lambda_p^\alpha$ and let

$$\beta = (\lambda_1^\beta, \lambda_2^\beta, \ldots, \lambda_q^\beta)$$

where $\lambda_1^\beta \geq \lambda_2^\beta \cdots \geq \lambda_q^\beta$. If $\lambda_i^\alpha - \lambda_i^\beta = 0$ for all i and $p = q$ then $\alpha \textcircled{=} \beta$. If the first non-zero difference $\lambda_i^\alpha - \lambda_i^\beta$ for $i = 1, 2, \ldots$ is negative then $\alpha \textcircled{<} \beta$.

LEMMA 3.11. *If $\alpha \textcircled{<} \beta$ then*

$$e_g^\alpha e_h^\beta = 0$$

for all $g, h \in S_N$.

PROOF: The proof is omitted.[5]

COROLLARY. *If $\alpha \textcircled{<} \beta$ then $e_h^\beta e_g^\alpha$ is nilpotent for all $g, h \in S_N$.*

PROOF: $(e_h^\beta e_g^\alpha)^2 = e_h^\beta (e_g^\alpha e_h^\beta) e_g^\alpha = 0$
by Lemma 3.11.

Young showed that for $\alpha \circledless \beta$ there exists a transposition

$$t \in H_h^\beta \cap V_g^\alpha.$$

It is this fact that is responsible for Lemma 3.11 and its corollary. Moreover, it permits generalization of Lemmas 3.7 and 3.8 to the case of two cycle structures. The proofs to the generalizations are almost identical with those to Lemmas 3.7 and 3.8 Therefore, it follows that

LEMMA 3.12. *If $\alpha \circledless \beta$ there exists $l, k \in S_N$ such that*

$$e_k^\beta e_l^\alpha = \pm e_h^\beta e_g^\alpha$$

for each pair $g, h \in S_N$.

PROOF: The proof follows directly from that to Lemma 3.7 .

LEMMA 3.13. *If $\alpha \circledless \beta$ define $P_{hg}^{\beta\alpha}$ and $N_{hg}^{\beta\alpha}$ by*

$$P_{hg}^{\beta\alpha} = \{e_k^\beta e_l^\alpha \quad \text{for } l, k \in S_N \mid e_k^\beta e_l^\alpha = + e_h^\beta e_g^\alpha\}$$

and

$$N_{hg}^{\beta\alpha} = \{e_k^\beta e_l^\alpha \quad \text{for } l, k \in S_N \mid e_k^\beta e_l^\alpha = - e_h^\beta e_g^\alpha\},$$

then

$$\nu(P_{hg}^{\beta\alpha}) = \nu(N_{hg}^{\beta\alpha}).$$

PROOF: Because a transposition $t \in H_h^\beta \cap V_g^\alpha$ exists, the proof to Lemma 3.8 carries over to this lemma with no essential changes.

Lemmas 3.12 and 3.13 permit the following important theorem which along with Theorem 3.1 forms the foundation to this entire theory.

THEOREM 3.2. *If $\alpha \circledless \beta$, then*

$$P^\alpha P^\beta = 0 \qquad \text{and} \qquad P^\beta P^\alpha = 0.$$

PROOF: By Lemma 3.10.

$$\sum_{g \in S_N} e_g^\alpha \sum_{h \in S_N} e_h^\beta = \sum_{g \in S_N} \sum_{h \in S_N} e_g^\alpha e_h^\beta$$

$$= 0.$$

Therefore, $P^\alpha P^\beta = 0$. Going the other way

$$\sum_{g\in S_N} e_g{}^\beta \sum_{h\in S_N} e_h{}^\alpha = \sum_{g\in S_N} \sum_{h\in S_N} e_g{}^\beta e_h{}^\alpha.$$

The right-hand side of the above expression contains only nilpotent terms according to the corollary to Lemma 3.11. However, Lemma 3.13 shows that they all must add up to zero. The proof of this remark mirrors that used in the proof to Theorem 3.1 to eliminate the nilpotent terms. Therefore, $P^\beta P^\alpha = 0$ also. This completes the proof.

There are as many P^α's as there are conjugate classes in S_N. Theorems 3.1 and 3.2 show that the P^α's are mutually orthogonal idempotents of simple two-sided ideals in $R(S_N)$. The P^α's generate the irreducible inequivalent characters of S_N.

4. More Relationships

In the following, the discussion will be restricted to arbitrary but fixed cycle structure, α. Presently, P^α is given by

$$P^\alpha = \frac{1}{\omega^\alpha} a^\alpha = \frac{1}{\omega^\alpha} \sum_{g\in S_N} e_g{}^\alpha.$$

The definition of $e_g{}^\alpha$ was

$$e_g{}^\alpha = \frac{1}{\lambda^\alpha} S_g{}^\alpha A_g{}^\alpha.$$

Lemma 4.1. *For $g \neq h$ either*

$$S_g{}^\alpha A_h{}^\alpha = 0$$

or there exists $l \in S_N$ such that

$$S_l{}^\alpha A_l{}^\alpha = S_g{}^\alpha A_h{}^\alpha.$$

Proof: Either $H_g{}^\alpha \cap V_h{}^\alpha \neq \{e\}$ or $H_g{}^\alpha \cap V_h{}^\alpha = \{e\}$. If $H_g{}^\alpha \cap V_h{}^\alpha \neq \{e\}$ then Lemma 3.2 guarantees the existence of a transposition $t \in H_h{}^\alpha \cap V_h{}^\alpha$. Therefore

$$S_g{}^\alpha t = S_g{}^\alpha \quad \text{and} \quad tA_h{}^\alpha = -A_h{}^\alpha,$$

from which it follows that

$$S_g{}^a A_h{}^a = S_g{}^a t A_h{}^a$$
$$= - S_g{}^a A_h{}^a = 0.$$

If $H_g{}^a \cap V_h{}^a = \{e\}$, then Lemma 3.5 guarantees that

$$T_h{}^a = pqT_g{}^a,$$

where $p \in H_g{}^a$ and $q \in V_g{}^a$. However,

$$pq = pqp^{-1}p.$$

Let $\bar{q} = pqp^{-1}$. Therefore,

$$T_h{}^a = \bar{q}pT_g{}^a.$$

Young showed that under these circumstances $\bar{q} \in V_h{}^a$. Therefore,

$$\bar{q}^{-1}T_h{}^a = pT_g{}^a,$$

where $\bar{q}^{-1} \in V_h{}^a$ and $p \in H_g{}^a$. Denote $\bar{q}^{-1}T_h{}^a(= pT_g{}^a)$ by $T_l{}^a$. Therefore, by Theorem 2.1

$$V_l{}^a = V_h{}^a \qquad \text{and} \qquad H_l{}^a = H_g{}^a.$$

This implies

$$A_l{}^a = A_h{}^a \qquad \text{and} \qquad S_l{}^a = S_g{}^a,$$

which gives

$$S_l{}^a A_l{}^a = S_g{}^a A_h{}^a.$$

LEMMA 4.2. *Define* $M_g{}^a$ *by*

$$M_g{}^a = \{S_l{}^a A_k{}^a \qquad \text{for } l, k \in S_N \mid S_l{}^a A_k{}^a = S_g{}^a A_g{}^a\}.$$

For any pair $g, h \in S_N$

$$\nu(M_g{}^a) = \nu(M_h{}^a).$$

PROOF: By Theorem 2.1

$$e_g{}^a = se_h{}^a s^{-1},$$

where $s = gh^{-1}$. Therefore,

$$S_g{}^a A_g{}^a = sS_h{}^a A_h{}^a s^{-1}.$$

Suppose that $S_l^a A_k^a \in M_h^a$. Therefore,

$$S_l^a A_k^a = S_h^a A_h^a$$

from which it follows that

$$\begin{aligned} sS_h^a A_h^a s^{-1} &= sS_l^a A_k^a s^{-1} \\ &= sS_l^a s^{-1} s A_k^a s^{-1} \\ &= S_{sl}^a A_{sk}^a . \end{aligned}$$

Therefore,

$$S_{sl}^a A_{sk}^a = S_g^a A_g^a$$

which implies $S_{sl}^a A_{sk}^a \in M_g^a$. However, since conjugation by s produces a one-to-one mapping of M_h^a into M_g^a, then

$$\nu(M_h^a) \leq \nu(M_g^a).$$

The inverse inequality is obtained similarly, giving

$$\nu(M_h^a) \geq \nu(M_g^a),$$

from which the conclusion of the lemma follows

$$\nu(M_g^a) = \nu(M_h^a).$$

THEOREM 4.1. $(\sum_{g \in S_N} S_g^a)(\sum_{h \in S_N} A_h^a) = \eta^a \sum_{g \in S_N} S_g^a A_g^a$, *where* η^a *is some real number.*

PROOF: $(\sum_{g \in S_N} S_g^a)(\sum_{h \in S_N} A_h^a) = \sum_{g \in S_N} \sum_{h \in S_N} S_g^a A_h^a$. Each term of the right hand sum is either zero or equal to $S_l^a A_l^a$ for some $l \in S_N$ according to Lemma 4.1. Therefore,

$$\left(\sum_{g \in S_N} S_g^a\right) \left(\sum_{h \in S_N} A_h^a\right) = \sum_{g \in S_N} \eta_g^a S_g^a A_g^a,$$

where the η_g^a's are integers. From Lemma 4.2 it is seen that η_g^a is explicitly given by $\nu(M_g^a)$. Since $\nu(M_g^a)$ is independent of g, then

$$\eta_g^a = \eta^a$$

for all $g \in S_N$ where η^a is some integer. Therefore,

$$\left(\sum_{g \in S_N} S_g^a\right) \left(\sum_{h \in S_N} A_h^a\right) = \eta^a \sum_{g \in S_N} S_g^a A_g^a.$$

Theorem 4.1 permits rewriting P^α as

$$P^\alpha = (\eta^\alpha \omega^\alpha \lambda^\alpha)^{-1} \Big(\sum_{g \in S_N} S_g{}^\alpha \Big) \Big(\sum_{h \in S_N} A_h{}^\alpha \Big).$$

$S_g{}^\alpha$ was defined in Section 2 by

$$S_g{}^\alpha = \sum_{s \in H^\alpha{}_g} s \, .$$

Since $H_g{}^\alpha$ is determined by a tableaux constructible from a frame associated with the cycle structure α, then the cycle structure of any element of $H_g{}^\alpha$ is either α or a refinement of α. The partial order, $<$, on the set of cycle structures is defined by: $\beta < \alpha$ iff β is a refinement of α. Therefore, for any $s \in H_g{}^\alpha$, the cycle structure of s, β, satisfies $\beta \leq \alpha$.

Given a conjugate class of S_N, C_β, with $\beta \leq \alpha$, only some of the elements of C_β are necessarily contained in a given $H_g{}^\alpha$ for $g \in S_N$. Other elements of C_β may be found in other $H_h{}^\alpha$'s with $h \in S_N$. H^α is defined to be the set obtained by taking together all the elements of each $H_g{}^\alpha$ for $g \in S_N$ with the proviso that, if an element $s \in S_N$ is contained in m of the $H_g{}^\alpha$'s, then it is contained in H^α m times. Therefore

$$\sum_{g \in S_N} S_g{}^\alpha = \sum_{g \in S_N} \sum_{s \in H^\alpha{}_g} s = \sum_{s \in H^\alpha} s.$$

Let $m_g^{\beta\alpha}$ denote the number of times the element $g \in C_\beta$ for $\beta \leq \alpha$ is contained in H^α.

LEMMA 4.3. *For $g, h \in C_\beta$ with $g \neq h$*

$$m_g^{\beta\alpha} = m_h^{\beta\alpha}.$$

PROOF: g is contained in $m_g^{\beta\alpha}$ different $H_s{}^\alpha$'s for $s \in S_N$. Denote the $H_s{}^\alpha$'s which contain g by

$$H_{s_1}^\alpha, H_{s_2}^\alpha, \ldots, H_{s_p}^\alpha$$

where $p = m_g^{\beta\alpha}$. Since $g, h \in C_\beta$, there exists $s \in S_N$ such that

$$h = s\, g s^{-1}.$$

Therefore, by conjugation, $g \in H_{s_i}^\alpha$ for $i = 1, 2, \ldots, p = m_g^{\beta\alpha}$ implies

$$sgs^{-1} = h \in sH_{s_i}^\alpha s^{-1} = H_{ss_i}^\alpha ,$$

from which it can be concluded that

$$m_g^{\beta\alpha} \leq m_h^{\beta\alpha}.$$

A similar argument in reverse order gives

$$m_h^{\beta\alpha} \leq m_g^{\beta\alpha},$$

from which the lemma follows:

$$m_g^{\beta\alpha} = m_h^{\beta\alpha}.$$

From Lemma 4.3 it is seen that H^α contains the entire conjugate class C_β for $\beta \leq \alpha$ $m^{\beta\alpha}$ times. Theorem 2.1 can be used to show that the number of elements of C_β for $\beta \leq \alpha$ in $H_g{}^\alpha$ is independent of g. This number is denoted by $N(\alpha, \beta)$. (The number of elements in C_β is denoted by m_β.) H^α is comprised of elements from $N!$ $H_g{}^\alpha$'s. Therefore,

$$m^{\beta\alpha} = \frac{N!N(\alpha, \beta)}{m_\beta}.$$

Define V^α as the set obtained by taking together all the elements of each $V_g{}^\alpha$ for $g \in S_N$ with the proviso that if an element $s \in S_N$ is contained in $\bar{m}$ of the $V_g{}^\alpha$'s, then it is contained in V^α $\bar{m}$ times. Therefore,

$$\sum_{g\in S_N} A_g{}^\alpha = \sum_{g\in S_N} \sum_{s\in V^\alpha{}_g} \varepsilon_s s = \sum_{s\in V^\alpha} \varepsilon_s s.$$

Let $\bar{m}_g^{\beta\alpha}$ denote the number of times the element $g \in C_\beta$ for $\beta \leq \bar{\alpha}$ is contained in V^α. Lemma 4.3 trivially generalizes to this case also and gives

$$\bar{m}_g^{\beta\alpha} = \bar{m}_h^{\beta\alpha}$$

for $g, h \in C_\beta$ with $g \neq h$. Therefore, V^α contains the entire conjugate class C_β for $\beta \leq \bar{\alpha}$ $\bar{m}^{\beta\alpha}$ times. Theorem 2.1 also implies that the number of elements of C_β for $\beta \leq \bar{\alpha}$ in $V_g{}^\alpha$ is independent of g. This number is denoted by $N(\bar{\alpha}, \beta)$. Since there are $N!$ $V_g{}^\alpha$'s these remarks guarantee

$$\bar{m}^{\beta\alpha} = \frac{N!N(\bar{\alpha}, \beta)}{m_\beta}.$$

LEMMA 4.4.

$$\sum_{g\in S_N} S_g{}^\alpha = \sum_{\beta\leq\alpha} \frac{N!}{m_\beta} N(\alpha, \beta) C_\beta$$

and

$$\sum_{g\in S_N} A_g{}^{\alpha} = \sum_{\beta'\leq\bar{\alpha}} \frac{N!}{m_\beta} N(\bar{\alpha}, \beta')\varepsilon_{\beta'}C_{\beta'} ,$$

where C_β denotes the ring element obtained from the conjugate class C_β by adding all its elements together and $\varepsilon_{\beta'}$ is plus one if β' is even and minus one if β' is odd.

PROOF: The remarks of the preceding paragraphs give

$$\sum_{g\in S_N} S_g{}^{\alpha} = \sum_{s\in H^\alpha} s = \sum_{\beta\leq\alpha} m^{\beta\alpha}C_\beta$$

$$= \sum_{\beta\leq\alpha} \frac{N!}{m_\beta} N(\alpha, \beta)C_\beta$$

and

$$\sum_{g\in S_N} A_g{}^{\alpha} = \sum_{s\in V^\alpha} \varepsilon_s s = \sum_{\beta'\leq\bar{\alpha}} \bar{m}^{\beta'\bar{\alpha}}\varepsilon_{\beta'}C_{\beta'}$$

$$= \sum_{\beta'\leq\bar{\alpha}} \frac{N!}{m_{\beta'}} N(\bar{\alpha}, \beta')\varepsilon_{\beta'}C_{\beta'} .$$

Define S^α and A^α by

$$S^\alpha = \sum_{g\in S_N} S_g{}^{\alpha} \quad \text{and} \quad A^\alpha = \sum_{g\in S_N} A_g{}^{\alpha}.$$

Therefore, P^α has been shown to equal

$$P^\alpha = (\eta^\alpha\omega^\alpha\lambda^\alpha)^{-1}S^\alpha A^\alpha.$$

THEOREM 4.2. *$P^{\alpha'}S^\alpha = 0$ if $\alpha' \otimes \alpha$.*

PROOF. $P^{\alpha'} = (\eta^{\alpha'}\omega^{\alpha'}\lambda^{\alpha'})^{-1}S^{\alpha'}A^{\alpha'}$. Therefore, $P^{\alpha'}S^\alpha = (\eta^{\alpha'}\omega^{\alpha'}\lambda^{\alpha'})^{-1} S^{\alpha'}$ $(\sum_{g\in S_N} S_g^{\alpha'})$ $(\sum_{h\in S_N} S_h^{\alpha})$. In the discussion following Lemma 3. 11 it was shown that for $\alpha' \otimes \alpha$ there exists a transposition $t \in H_h{}^{\alpha} \cap V_g^{\alpha'}$ for each pair of g and h. Therefore, $A_g^{\alpha'}S_h{}^{\alpha} = 0$ for all g and h. Therefore, $P^{\alpha'}S^\alpha = 0$.

5. THE FUNDAMENTAL THEOREM

Frobenius defined the compound characters of the symmetric groups by

$$\Phi^\alpha(C_\sigma) = \frac{N!}{h_\alpha} \frac{N(\alpha, \sigma)}{m_\sigma} .$$

α and σ both label cycle structures and conjugate classes since the two are isomorphic for the symmetric groups.[7] $N!$ is the order of the symmetric group S_N. C_σ is the conjugate class associated with cycle structure σ. $\Phi^\alpha(C_\sigma)$ means the α-th compound character evaluated on any element of C_σ. Write α as $\alpha = (\lambda_1, \lambda_2, \ldots \lambda_l)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Then $h_\alpha = \lambda_1!\, \lambda_2! \cdots \lambda_l!$. m_σ is the number of elements of S_N contained in the conjugate class C_σ. Given $\alpha = (\lambda_1, \ldots, \lambda_l)$ where $\lambda_1 \geq \cdots \geq \lambda_l$, consider $G_{(\alpha)} \equiv S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_l}$ where S_{λ_i} is the symmetric group on λ_i elements. $N(\alpha, \sigma)$ counts how many elements of conjugate class $C_\sigma \subset S_N$ are contained in $G_{(\alpha)}$. A necessary and sufficient condition for $N(\alpha, \sigma) = 0$ is that the cycle structure σ is a refinement of the cycle structure α. The cycle structure σ can also be written as $\sigma = (l_1^{m_1}, l_2^{m_2}, \ldots, l_k^{m_k})$ where $l_1 > l_2 > \cdots > l_k$ and the m_i's are the multiplicities of the l_i's. Writing σ in this way and writing α as $\alpha = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ permits writing the explicit formula for $N(\alpha, \sigma)$ as

$$N(\alpha, \sigma) = \sum_{m_{ij}}^{\beta} \prod_{i}^{\alpha} \frac{\lambda_i!}{\prod\limits_{j}^{\beta} l_j^{m_{ji}}\, m_{ij}!}$$

where $\sum_i^\beta m_{ji} = m_j$ and $\sum_j^\beta m_{ji} l_j = \lambda_i$.[8] With all these definitions and formulae the compound characters

$$\Phi^\alpha(C_\sigma) = \frac{N!}{h_\alpha} \frac{N(\alpha, \sigma)}{m_\sigma}$$

may be computed readily.

Since the $\Phi^\alpha(C_0)$'s are compound characters they may be written as an expansion in terms of the irreducible characters

$$\Phi^\alpha(C_\sigma) = \sum_{\alpha'} a_{\alpha'}^{\alpha} \chi^{\alpha'}(C_\sigma),$$

where $\chi^{\alpha'}(C_0)$ denotes the α'-th irreducible character evaluated on class C_σ and where the sum is taken over all cycle structures α'. The $a_{\alpha'}^{\alpha}$'s are integers. The orthogonality relations for the irreducible characters give

$$a_{\alpha'}^{\alpha} = \frac{1}{N!} \sum_{\sigma} m_\sigma\, \Phi^\alpha(C_\sigma) \chi^{\alpha'}(C_\sigma).$$

[7] These defining remarks for $\Phi^\alpha(C_\sigma)$ may be found in [1] and [2].

[8] The α's and β's over Σ and Π symbols indicate the cycle structures to which the arguments of Σ and Π refer.

Using the definition of $\Phi^{\alpha}(C_\sigma)$ the above expression becomes

$$a^{\alpha}_{\alpha'} = \frac{1}{h_\alpha} \sum_{\sigma} N(\alpha, \sigma)\chi^{\alpha'}(C_\sigma).$$

On the set of cycle structures of S_N a linear order and a partial order were defined in Sections III and IV. Using these definitions we have the following fundamental theorem.

THEOREM 5.1. $\sum_{\sigma \leq \alpha} N(\alpha, \sigma)\chi^{\alpha'}(C_\sigma) = 0$ *if* $\alpha' \circledless \alpha$.

PROOF: Consider the group ring $R(S_N)$ generated by S_N. Define the ring element C_β by

$$C_\beta = \sum_{g \in C_\beta} g$$

where the sum is taken over all elements of conjugate class C_β. It is well known [3] that

$$P^{\alpha} = \frac{1}{N!} \chi^{\alpha}(e) \sum_{\beta} \chi^{\alpha}(C_\beta)C_\beta$$

where P^{α} is the idempotent of $R(S_N)$ which generates the simple two-sided ideal of $R(S_N)$, which corresponds with the α-th irreducible representation of S_N. The formula may be inverted to give

$$C_\beta = \sum_{\sigma} \frac{1}{\chi^{\sigma}(e)} \chi^{\sigma}(C_\beta)m_\beta P^{\sigma}.$$

In Section IV the quantity

$$S^{\alpha} = \sum_{\beta \leq \alpha} \frac{N!}{m_\beta} N(\alpha, \beta)C_\beta$$

is shown to satisfy

$$P^{\alpha'}S^{\alpha} = 0 \qquad \text{if } \alpha' \circledless \alpha.$$

Using the inversion formula above, S^{α} may be written as

$$S^{\alpha} = \sum_{\sigma} \sum_{\beta \leq \alpha} \frac{N!}{\chi^{\sigma}(e)} N(\alpha, \beta)\chi^{\sigma}(C_\beta)P^{\sigma}.$$

The idempotents, P^{α}, satisfy $P^{\alpha}P^{\alpha'} = P^{\alpha}\delta_{\alpha,\alpha'}$, where $\delta_{\alpha,\alpha'}$ is the Kronecker delta. Therefore,

$$P^{\alpha'}S^{\alpha} = \sum_{\sigma} \frac{N!}{\chi^{\sigma}(e)} \sum_{\beta \leq \alpha} N(\alpha, \beta)\chi^{\sigma}(C_{\beta})P^{\alpha'}P^{\sigma}$$

$$\frac{N!}{\chi^{\alpha'}(e)} \sum_{\beta \leq \alpha} N(\alpha, \beta)\chi^{\alpha'}(C_{\beta})P^{\alpha'}.$$

However, for $\alpha' \circledless \alpha$, $P^{\alpha'}S^{\alpha} = 0$ implies that

$$\sum_{\beta \leq \alpha} N(\alpha, \beta)\chi^{\alpha'}(C_{\beta}) = 0 \qquad \text{if } \alpha' \circledless \alpha.$$

Since $N(\alpha, \sigma) = 0$ unless $\sigma \leq \alpha$ then the expression

$$a^{\alpha}_{\alpha'} = \frac{1}{h_{\alpha}} \sum_{\sigma} N(\alpha, \sigma)\chi^{\alpha'}(C_{\sigma})$$

can be rewritten as

$$a^{\alpha}_{\alpha'} = \frac{1}{h_{\alpha}} \sum_{\sigma \leq \alpha} N(\alpha, \sigma)\chi^{\alpha'}(C_{\sigma}).$$

The fundamental theorem shows that

$$a^{\alpha}_{\alpha'} = 0 \qquad \text{if } \alpha' \circledless \alpha.$$

If there are k conjugate classes, number them with the first k natural numbers such that if $\beta \circledless \alpha$ then $j_{\beta} < j_{\alpha}$ where j_{β} and j_{α} are the natural numbers used to number the classes. This can be done uniquely since $\circledless$ is a linear order on a finite set. The fundamental theorem gives

$$\begin{aligned} \Phi^{k} &= a_k{}^k\chi^k \\ \Phi^{k-1} &= a_k^{k-1}\chi^k + a_{k-1}^{k-1}\chi^{k-1} \\ &\vdots \\ \Phi^{1} &= a_k{}^1\chi^k + a_{k-1}^1\chi^{k-1} + \cdots + a_2{}^1\chi^2 + a_1{}^1\chi^1. \end{aligned}$$

At each successive stage one more irreducible character is introduced. The terms $a_j{}^j$ can be shown to be non-zero so that each stage does indeed introduce another irreducible character.

THEOREM 5.2. $a_j{}^j \neq 0$ *for all* j.

PROOF: If $a_j{}^j = 0$ then $P^jS^j = 0$ must follow. However, since $P^jP^j = P^j$ and $P^j = (\eta^j\omega^j\lambda^j)^{-1}S^jA^j$, then $P^jP^j = (\eta^j\omega^j\lambda^j)^{-1}P^jS^jA^j$. Therefore, P^jS^j cannot equal zero. Therefore, $a_j{}^j \neq 0$.

6. How to Calculate the Irreducible Characters

The triangular dependence shown above coupled with the orthogonality relations of the irreducible characters permits a trivial inversion process. The process is one of stage-by-stage elimination involving only addition and multiplication of numbers.

Notice that a_j^l is given by

$$a_j^l = \frac{1}{N!} \sum_\sigma m_\sigma \Phi^l(C_\sigma)\chi^j(C_\sigma).$$

Therefore,

$$\frac{1}{N!} \sum_\sigma m_\sigma \Phi^k(C_\sigma)\Phi^k(C_\sigma) = (a_k^k)^2$$

from which a_k^k is determined. χ^k is then given by $(1/a_k^k)\Phi^k$. Since χ^k is known

$$a_k^{k-1} = \frac{1}{N!} \sum_\sigma m_\sigma \Phi^{k-1}(C_\sigma)\chi^k(C_\sigma)$$

can be computed. This gives χ^{k-1} by

$$a_{k-1}^{k-1}\chi^{k-1} = \Phi^{k-1} - a_k^{k-1}\chi^k.$$

The coefficient a_{k-1}^{k-1} is computed in the same way as a_k^k. In general χ^k, χ^{k-1}, ... and χ^{k-1} will have been determined and they will give a_{k-i}^{k-j-1} by

$$a_{k-i}^{k-j-1} = \frac{1}{N!} \sum_\sigma m_\sigma \Phi^{k-j-1}(C_\sigma)\chi^{k-i}(C_\sigma)$$

for $i = 0, 1, 2, ..., j$. From these coefficients χ^{k-j-1} is given by

$$a_{k-j-1}^{k-j-1}\chi^{k-j-1} = \Phi^{k-j-1} - \sum_{i=0}^{j} a_{k-1}^{k-j-1}\, \chi^{k-1};$$

a_{k-j-1}^{k-j-1} is determined in the same way as a_k^k.

Since this inversion process is automatic the problem of obtaining the χ^k is only as difficult as writing down the Φ^k's, which is not difficult.

7. Problem

A further understanding of the significance of the Frobenius compound character formula would be gained if one understood the meaning of the $a_{\alpha}^{\alpha'}$ in some more basic manner. In particular, some purely combinatorial mechanism may be operative in determining the values of the $a_{\alpha}^{\alpha'}$. This is strongly suggested by the values one obtains for S_1, S_2, S_3, S_4, and S_5. For S_5 which has 7 conjugate classes the $a_{\alpha}^{\alpha'}$ table is:

$$a_{\alpha}^{\alpha'}: \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 1 & 3 & 3 & 3 & 2 & 1 & 0 \\ 1 & 4 & 5 & 6 & 5 & 4 & 1 \end{matrix}$$

Note that the first five rows are precisely the first five rows of the partition number table. The last row is easily intepreted as the dimensionalities of the irreducible representations of S_5. An interpretation for the sixth row is presently lacking. What can be said of the $a_{\alpha}^{\alpha'}$ table for S_N in general?

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