π in a Box
or
Easy as π

By

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My son was born on Pi-day back in 1974. On occasion I have written about Pi to commemorate his birth (and also Einstein's back in 1879). In my website you will find my account of how to get Pi exclusively using the Pythagorean theorem and the square-root of 2 (http://www.fefox.com/ARTICLES/PiDay.pdf). No trigonometric functions are used. At the time I thought that approach was novel but eventually discovered many antecedents of similar content.

On these pages, I present another context for Pi which is also not novel with me. My friend John Elton communicated the idea to me and sent me a YouTube website where the idea is animated (look up Pi and bouncing masses). There was a lot of activity with this problem in 2019. I have not looked at any of the associated analysis but instead have tried my hand at it on my own. John did the same thing and we both hit upon similar solutions. What is interesting is the different mathematical structures that come into play and some subtleties that are easily missed. The study of π in the context described here appears to have begun in 2003 with a paper by Gregory A. Galperin of Eastern Illinois University. A relevant URL is
https://www.maths.tcd.ie/~lebed/Galperin.%20Playing%20pool%20with%20pi.pdf

The setting for this discussion is a one-dimensional line, say, the x-axis. Located at $x = 0$ is an infinitely massive wall. To the right of the wall is a small mass of mass m that may as well be considered to be a point mass. To the right of the small mass is a large mass of mass M. The small mass will be bouncing off of the wall and the large mass and each such collision will be assumed to be perfectly elastic. That means that momentum and kinetic energy are conserved at each impact. For the small mass and the wall, this means that the small mass momentum will be reversed, and the wall will remain unchanged. Initially the large mass is moving to the left with some initial velocity and the small mass is at rest. Eventually the large mass will impact the small mass, sending it off to the left where it will rebound from the wall and hit the large mass as the small mass comes back from the wall. There will be another impact between the moving masses. Two possibilities arise. In the first, the second impact of the moving masses will send the small mass back to the wall from which it will rebound, and the large mass will continue to move to the left, albeit more slowly than before. Another impact will occur. Or in the case of the second possibility, the large mass will develop a motion to the right. If it does, subsequent impacts from the small mass will eventually cause the large mass to move to the right faster than the small mass can, and the two masses move off to the right-side infinity, never touching again. How many collisions does the small mass make with the large mass and the wall? That is the problem: count the collisions until there are no more.

To make the problem more concrete, imagine that M = 100^r and that m = 1 where r is an integer and we use arbitrary mass units. The counting problem has the solution: Integer part of $10^r \times \pi$. If r = 1, then 31 collisions are counted; if r = 2, then 314 collisions are counted; and if r = 10, then the count gives 31415926535. The last digit for the r = 10 case is 5, not the rounded-up value 6. The man-made option of rounding up or down the last digit is not considered by the physics of collisions.

It is worth noting now that the solution does not require locating the positions of the two masses at any time nor the length of time between events.

1-d physics of collisions

Two conservation laws govern the dynamics. They are the conservation of momentum and the conservation of kinetic energy. In obvious notation these laws are

$$mu' + MV' = mu + MV$$

$$m(u')^2 + M(V')^2 = mu^2 + MV^2$$

These laws (I am dropping the unnecessary ½'s for the energy equation) are written in the standard format that has the *before* velocities on the right-hand side and the *after* velocities on the left. In 1-d, there is another way to write these laws.

$$mu^2 - m(u')^2 = M(V')^2 - MV^2$$

$$mu - mu' = MV - MV'$$

in which all large mass properties are on the right and the small mass properties are on the left, whether *before* or *after*. Dividing the left-hand sides and the right-hand sides yields the *linear* equation independent of the masses

$$u + u' = V' + V$$

The problem has been reduced to two simultaneous linear equations (only possible in 1-d). Rewriting the equation above and the conservation of momentum equation, we arrive at the pair of equations

$$u' - V' = V - u$$

$$mu' + MV' = MV + mu$$

These equations strongly suggest a matrix notation
Eq(1)

$$\begin{pmatrix} 1 & -1 \\ m & M \end{pmatrix} \begin{pmatrix} u' \\ V' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ m & M \end{pmatrix} \begin{pmatrix} u \\ V \end{pmatrix}$$

It is never a good idea to use matrices in which some elements have one kind of units and the others have some other kind of units, in this case mass and dimensionless. We can clean this up by using new dynamical variable with the units $\sqrt{mass} \times velocity$, that are the same as square-root of kinetic energy. The new quantities are defined by

$$\begin{pmatrix} q \\ Q \end{pmatrix} \coloneqq \begin{pmatrix} \sqrt{m}u \\ \sqrt{M}V \end{pmatrix} = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix} \begin{pmatrix} u \\ V \end{pmatrix}$$

In Eq(1) above we make the replacements

$$\begin{pmatrix} 1 & -1 \\ m & M \end{pmatrix} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix}^{-1} \begin{pmatrix} q' \\ Q' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ m & M \end{pmatrix} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix}^{-1} \begin{pmatrix} q \\ Q \end{pmatrix}$$

Multiplying from the left and remembering that matrix multiplication is not commutative in general we get the transition equation for a collision

$$\begin{pmatrix} q' \\ Q' \end{pmatrix} = \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ m & M \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ m & M \end{pmatrix} \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{M} \end{pmatrix}^{-1} \begin{pmatrix} q \\ Q \end{pmatrix}$$

The inverse matrix is given by

$$\begin{pmatrix} 1 & -1 \\ m & M \end{pmatrix}^{-1} = \frac{1}{m+M} \begin{pmatrix} M & 1 \\ -m & 1 \end{pmatrix}$$

Multiplying everything yields

$$\begin{pmatrix} q' \\ Q' \end{pmatrix} = \frac{1}{m+M} \begin{pmatrix} m-M & 2\sqrt{mM} \\ 2\sqrt{mM} & M-m \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}$$

It will simplify things to introduce the mass ratio $\mu = m/M$. This gives us Eq(2)

$$\begin{pmatrix} q' \\ Q' \end{pmatrix} = \frac{1}{\mu+1} \begin{pmatrix} \mu-1 & 2\sqrt{\mu} \\ 2\sqrt{\mu} & 1-\mu \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}$$

After every collision, except for the last one, the small mass hits the wall and rebounds. This amounts to reversing its velocity, but not the velocity for the large mass. This effect is captured by multiplying Eq(2) from the left by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

This gives the governing equation for every pair of collision and rebound Eq(3)

$$\begin{pmatrix} q' \\ Q' \end{pmatrix} = \frac{1}{\mu+1} \begin{pmatrix} 1-\mu & -2\sqrt{\mu} \\ 2\sqrt{\mu} & 1-\mu \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}$$

Quaternions and trigonometric identities

Dynamical equations that are linear and in 2 dimensions (here q and Q) are often expressed in terms of *quaternions* as represented by the *Pauli matrices*

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

These are the square-roots of unity in 2-d. They do not commute but instead anti-commute

$$[\sigma_i, \sigma_j] := \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\epsilon_{kij}\,\sigma_k$$

in which ϵ_{kij} is the Levi-Civita symbol, and is defined by

$$\epsilon_{kij} = \begin{matrix} +1, kij\ is\ xyz;\ yzx;\ zxy \\ -1, kij\ is\ xzy;\ zyx;\ yxz \\ 0, otherwise \end{matrix}$$

and

$$\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i = 0$$

for i, j, k equal to x, y, z with i ≠ j. The first Pauli matrix, σ_0, is simply the 2-d identity matrix.

The application of the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

made in the preceding section is just $-\sigma_z$. What I am about to do, I could have done before applying $-\sigma_z$. However, that would be too soon. Doing it now, however, leads to great simplifications. Looking at Eq(3) we see that

$$\left(\frac{1-\mu}{1+\mu}\right)^2 + \left(\mp\frac{2\sqrt{\mu}}{1+\mu}\right)^2 = 1$$

Therefore, we can make the identifications

$$cos(\theta) = \frac{1-\mu}{1+\mu}\ , \quad sin(\theta) = \frac{2\sqrt{\mu}}{1+\mu}$$

wherein

$$\theta = arctan\left(\frac{2\sqrt{\mu}}{1-\mu}\right)$$

Eq(3) becomes
Eq(4)

$$\begin{pmatrix} q' \\ Q' \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} q \\ Q \end{pmatrix}$$

$$= \left(\cos(\theta)\,\sigma_0 - \sin(\theta) i\sigma_y\right) \begin{pmatrix} q \\ Q \end{pmatrix}$$

The simplicity of this equation is plain to see. In addition, the formula for n iterations of this mapping is also remarkably simple

$$\begin{pmatrix} q_n \\ Q_n \end{pmatrix} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}^n \begin{pmatrix} q_0 \\ Q_0 \end{pmatrix}$$

$$= \begin{pmatrix} cos(n\theta) & -sin(n\theta) \\ sin(n\theta) & cos(n\theta) \end{pmatrix} \begin{pmatrix} q_0 \\ Q_0 \end{pmatrix}$$

$$= \left(\cos(n\theta)\,\sigma_0 - \sin(n\theta) i\sigma_y\right) \begin{pmatrix} q \\ Q \end{pmatrix}$$

as follows from induction and the trigonometric identities generated by

$$e^{i\theta} e^{i(n-1)\theta} = e^{in\theta}$$

$$(\cos(\theta) + i\, sin(\theta))\left(\cos\big((n-1)\theta\big) + i\, sin((n-1)\theta)\right)$$

$$= \cos(\theta)\cos\big((n-1)\theta\big) - \sin(\theta)\sin\big((n-1)\theta\big)$$

$$+i\,(sin(\theta)\cos\big((n-1)\theta\big)+\cos(\theta)\,sin((n-1)\theta))$$

One can also use the Pauli matrix version of Eq(4) and the properties of the 2-d square-root of "minus one" that is written as $i\sigma_y$. Finally, for the initial condition given earlier and formulized by

$$\begin{pmatrix} q_0 \\ Q_0 \end{pmatrix} = \begin{pmatrix} 0 \\ -s \end{pmatrix}$$

we obtain the explicit solution

$$\begin{pmatrix} q_n \\ Q_n \end{pmatrix} = \begin{pmatrix} \sin(n\theta)\,s \\ -\cos(n\theta)\,s \end{pmatrix}$$

Counting collisions

The angle θ is determined by μ and μ is determined by the mass ratio that is chosen. These quantities are fixed so that any condition we might impose as part of our analysis must respect these facts. In the first quadrant of a circle used to represent $n\theta$ both the sine and the cosine are positive. After a collision and a rebound we see that q is moving to the right and that Q is moving to the left so that another collision-rebound pair will occur. This happens everywhere in the quadrant (0, π/2). When the $n\theta$ values are large enough to be in the second quadrant (π/2, π), the sine is still positive, but the cosine is now negative. The solution above describes this situation as q still moving to the right but Q is also moving to the right. Where this new behavior starts, near π/2, the sine is large, and the cosine is small. So, the q velocity (u) is greater than the Q velocity (V) and we continue to get collision-rebound pairs. This continues with increasing n until u is no longer faster than V and V gets away to the right without any more collisions. The extremely large mass ratios we are considering imply, as we will show, that the exchange of the title, *the speedier*, occurs very close to π. It is tempting to set u and V equal and to solve for n but the likelihood of getting exactly an integer is remote for the reasons used to begin this paragraph. Instead we must use inequalities.

Assume there is an N such that $u_N > V_N > 0$ and $V_{N+1} > u_{N+1} > 0$. This is the exchange of the speedier we seek. Thus

$$\begin{pmatrix} u_N \\ V_N \end{pmatrix} = \begin{pmatrix} 1/\sqrt{m} & 0 \\ 0 & 1/\sqrt{M} \end{pmatrix} \begin{pmatrix} q_N \\ Q_N \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{\sin(N\theta)}{\sqrt{m}} \\ \dfrac{-\cos(N\theta)}{\sqrt{M}} \end{pmatrix} s$$

Similarly,

$$\begin{pmatrix} u_{N+1} \\ V_{N+1} \end{pmatrix} = \begin{pmatrix} \dfrac{\sin((N+1)\theta)}{\sqrt{m}} \\ \dfrac{-\cos((N+1)\theta)}{\sqrt{M}} \end{pmatrix} s$$

The inequalities, near π, become

$$\frac{\sin(N\theta)}{\sqrt{m}} > \frac{-\cos(N\theta)}{\sqrt{M}} > 0$$

$$\frac{-\cos((N+1)\theta)}{\sqrt{M}} > \frac{\sin((N+1)\theta)}{\sqrt{m}} > 0$$

To get a handle on N, I will now consider it to be continuous and simply solve for N*.

$$\frac{\sin(N^*\theta)}{\sqrt{m}} = \frac{-\cos(N^*\theta)}{\sqrt{M}}$$

Therefore

$$N^* = -\frac{1}{\theta} arctan\left(\sqrt{\mu}\right) = -\frac{arctan(\sqrt{\mu})}{arctan\left(\dfrac{2\sqrt{\mu}}{1-\mu}\right)}$$

which for small μ, such as 1/10,000, is close to $-1/2$. Blindly dealing with tangents (and arctangents) near π can get you into trouble (we expect N^* to be large and positive). Let us assume that $N^*\theta$ is close to π. We will write $N^*\theta = \pi - \epsilon$ where ϵ is small in some sense (to be determined). The equality condition is now

$$\frac{sin(\pi-\epsilon)}{\sqrt{m}} = \frac{-cos(\pi-\epsilon)}{\sqrt{M}}$$

$$\frac{\sin(\pi)\cos(-\epsilon) + cos(\pi)sin(-\epsilon)}{\sqrt{m}} = \frac{-cos(\pi)cos(-\epsilon) + sin(\pi)sin(-\epsilon)}{\sqrt{M}}$$

$$\frac{sin(\epsilon)}{\sqrt{m}} = \frac{cos(\epsilon)}{\sqrt{M}}$$

$$\epsilon = arctan(\sqrt{\mu})$$

This makes perfect sense. When μ is small so is ϵ. Moreover, for small μ we obtain the condition $\epsilon < \theta$, or put even more strongly, $\epsilon \approx \theta/2$. Not only did we find where the exchange of the speedier occurs, but it is the last possible integer for $N\theta < \pi$. N is the integer part of N^*, and $N+1$ puts the angle $(N+1)\theta$ into the third quadrant where no more collisions or rebounds can occur.

Let us summarize what we have at this point. The mass ratio is $\mu = m/M$. Usually, we choose $\mu = 100^{-r}$ in which r is an integer. The mapping of the velocities from after a collision-rebound pair to after the next collision-rebound pair of velocities is expressed in terms of the finite angle θ which is given in terms of the mass ratio by

$$\theta = arctan\left(\frac{2\sqrt{\mu}}{1-\mu}\right) \approx 2\sqrt{\mu}$$

where the approximation is good to 3 parts in 200, for $r = 1$. The more precise value for $r = 1$ is .19933 radians. The integer part (IP) of dividing π by θ is N. Therefore

$$N = IP(\pi/\theta) = IP(\pi/.19933) = IP\big((\pi/1.9933)\times 10\big)$$

$$= IP(15.76) = 15$$

Each θ event corresponds with 2 collisions, one between the two masses and one with the wall. This leads to
Eq(5)

$$2N = 2 \times IP\left(\frac{\pi}{2} \times 10^r\right)$$

Since multiplying by 2 and taking the Integer Part do not commute and because the digits of π are not all even, we are not finished. For example, let $r = 3$ and consider Eq(5).

$$2N = 2 \times IP(1.5707 \ldots \times 10^3) = 3140$$

We are short one collision, but **not** a collision-rebound pair. Is it possible that the last collision has the property that both velocities are positive with the large mass moving faster than the small mass? After N events the state of the system is given by

$$\begin{pmatrix} q_N \\ Q_N \end{pmatrix} = \begin{pmatrix} sin(N\theta) \\ -cos(N\theta) \end{pmatrix} s$$

For $r = 3$, $\theta = .00199999933$ radians and $2N = 3140$. Dropping the s above we have

$$\begin{pmatrix} q_N \\ Q_N \end{pmatrix} = \begin{pmatrix} sin(3.139998) \\ -cos(3.139998) \end{pmatrix} = \begin{pmatrix} .001594 \\ .999998 \end{pmatrix}$$

The corresponding velocities are

$$\begin{pmatrix} u_N \\ V_N \end{pmatrix} = \begin{pmatrix} 0.001594 \\ 0.000999 \end{pmatrix}$$

What happens to this state? If we apply the matrix

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

we automatically also include another wall rebound. The correct application for just a two-mass collision is

$$\begin{pmatrix} -cos(\theta) & sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

Using the Nth (q, Q) state from above we get

$$\begin{pmatrix} -cos(\theta) & sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \begin{pmatrix} .001594 \\ .999998 \end{pmatrix} = \begin{pmatrix} .00040 \\ .999999 \end{pmatrix}$$

where the value $\theta = .00199999933$ is used. This is for (q, Q). To get (u, V) we need to divide the q value by 1 and the Q value by 10^3. The result for the velocities is

$$\begin{pmatrix} u_{N\prime} \\ V_{N\prime} \end{pmatrix} = \begin{pmatrix} .00040 \\ .00099 \end{pmatrix}$$

This is the last event, a single collision followed by both masses moving to the right, with no wall rebound and with the Q mass the speedier.

As r is increases, the finite number of digits of π also increases. If the last digit of this finite set of digits is odd, then the circumstances needed for one last collision between the masses after which both masses continue to move to the right will be realized. The example of this given above is generic. On the other hand, if the last digit is even, then the last event will be a collision-rebound pair in which the small mass recoils off of the large mass and then rebounds from the wall but never catches up to the large mass again. This is also generic. Note that whichever happens, it is the next digit in the sequence of digits of π that is the last digit, not a rounded off value.