Stochastic resonance in a double well

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A theoretical explanation for the phenomenon of *stochastic resonance* in a double well is presented. Starting with the standard Langevin description for this process, a systematic, time-dependent, perturbation-theory analysis of the associated Fokker-Planck equation yields stochastic resonance. In order to get stochastic resonance, the governing parameters must satisfy special inequalities. The analysis also produces these inequalities.

INTRODUCTION

Recently, McNamara, Wiesenfeld, and Roy performed an interesting experiment[1] with a ring laser in which they observed *stochastic resonance.* This phenomenon is characterized by the enhancement of the signal-to-noise ratio caused by injection of noise into a periodically modulated nonlinear system. As the noise strength increases from initially small values, the signal-to-noise ratio achieves a maximum before the inevitable decay for large values sets in. A greater than twofold enhancement is seen in the signal-to-noise ratio. Only one other experiment,[2] on a two-state Schmitt trigger circuit, has been reported previously. The enhancement profile has also been seen in numerical simulation.[3]

Benzi and co-workers[4–6] introduced the concept in order to explain the periodicity of the Earth's ice ages. Their highly phenomenological theory captured the essence of the effect.

A more detailed theory[7–9] of the Fokker-Planck description, including periodic modulation handled adiabatically, has bolstered understanding of the process. This particular approach *contracts the description* down to the populations in each of the two wells. The signal-to-noise enhancement profile is seen when the spectral frequency, ω, is set equal to Ω, the modulation frequency. This is the only sense in which the appellation *resonance* is appropriate. As we shall see below, there is no resonance between the mean hopping rate and the modulation frequency. At the modulation frequency, the signal-to-noise ratio deduced from the power spectrum has an enhancement profile

$$\frac{S}{N}=\frac{C}{D^2}\exp\left[-\frac{\Delta U}{D}\right] , \tag{1}$$

in which C is a constant, D measures the noise strength (proportional to mean square), and ΔU measures the barrier height between the two wells (symmetric well depth case). (In Ref. 9 there is a factor of 2 difference in the definition of D compared with here.)

In this paper we present an account of the Fokker-Planck theory without resort to further contraction of the description. We do so through a systematic analysis of the eigenfunction-eigenvalue problem for the unmodulated Fokker-Planck equation, and of the time-dependent perturbation theory required for the periodically modulated Fokker-Planck equation. The origin of Eq. (1) for the contracted theory is explained. Inequalities necessary for the conditions exhibiting stochastic resonance are derived.

MATHEMATICAL MODEL FOR STOCHASTIC RESONANCE

The standard model for this phenomenon is given by one-dimensional, overdamped motion in a double well:

$$\dot{x}=ax-bx^3+g_w+a_0\cos(\Omega t+\gamma) , \tag{2}$$

in which x is the position (*position* is meant figuratively; for the ring laser the laser electric field is signified by x), g_w is Gaussian white noise, a_0 is the modulation amplitude, Ω is the modulation frequency, and γ is the random phase of modulation. When the equation above is interpreted as representing overdamped motion in a double-well potential, the potential contemplated is

$$U(x)=-\frac{a}{2}x^2+\frac{b}{4}x^4 , \tag{3}$$

with minimum at $\pm(a/b)^{1/2}$ and an intervening barrier of relative height $a^2/4b$. The Gaussian white noise is characterized by its first two moments

$$\langle g_w(t)\rangle=0 , \tag{4}$$

$$\langle g_w(t)g_w(s)\rangle=2D\delta(t-s) , \tag{5}$$

in which D is the noise strength. [In Ref. 9 there is no 2 in (5).] Note that if x has units of length then both D and $U(x)$ have the units of a diffusion constant (cm^2/sec). The random phase is uniformly distributed on the interval $[0,2\pi]$.

In the absence of modulation ($a_0\equiv 0$), the Langevin equation (2) immediately leads[10] to an associated Fokker-Planck equation for a stationary Markov process:

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial x}[(ax-bx^3)P]+D\frac{\partial^2}{\partial x^2}P , \tag{6}$$

where $P=P(x,t)$ is the probability distribution for x. The presence of the modulation ($a_0\neq 0$), however, destroys stationarity. Stationarity is restored by averaging over the random phase, producing an equation for the γ-averaged P, to be denoted by $\bar{P}$. Nevertheless, the process is no longer Markovian on account of the modulation, which means that the equation for $\bar{P}$ is not a bona fide Fokker-Planck equation. It is still the equation for the probability distribution, and it may be solved for $\bar{P}$ anyway, at least for sufficiently small a_0.

Because γ is random and a_0 will be taken as small, the $a_0\cos(\Omega+\gamma)$ in (2) may be viewed as a small random perturbation. The probability distribution equation obtained from the g_w fluctuations when the a_0 term is present is a simple generalization of (6):

$$\frac{\partial}{\partial t}P=-\frac{\partial}{\partial x}[(ax-bx^3)P]-\frac{\partial}{\partial x}[a_0\cos(\Omega t+\gamma)P]+D\frac{\partial^2}{\partial x^2}P\ . \tag{7}$$

This probability distribution equation is itself a random equation because of the γ fluctuations. In fact, it is in the class of multiplicative stochastic Langevin equations. Averaging over γ by means of the cumulant expansion,[10] which respects the noncommutativity of the differential operators on the right-hand side of (7), we obtain the second cumulant approximation (see the Appendix):

$$\frac{\partial}{\partial t}\bar{P}=-\frac{\partial}{\partial x}[(ax-bx^3)\bar{P}]+D\frac{\partial^2}{\partial x^2}\bar{P}+\tfrac{1}{2}a_0^2\int_0^t ds\,\cos[\Omega(t-s)]\frac{\partial}{\partial x}\exp[(t-s)L]\times\frac{\partial}{\partial x}\exp[(s-t)L]\bar{P}\ , \tag{8}$$

in which L is an operator defined by

$$L\equiv-\frac{\partial}{\partial x}(ax-bx^3)+D\frac{\partial^2}{\partial x^2}\ . \tag{9}$$

This description presents a formidable mathematical problem. Even for $a_0=0$, the bona fide Fokker-Planck equation (6) does not yield closed-form expressions for its eigenfunctions and eigenvalues. Nevertheless, much is known about their general properties,[11] both analytically and numerically. These properties will enable us to find a formal representation for stochastic resonance on the basis of Eq. (8).

EIGENFUNCTION EXPANSIONS

The smallness of a_0 also allows us to handle Eq. (8) with time-dependent perturbation theory. To do this, we need the solution to the unperturbed problem, the Fokker-Planck equation (6). It has the steady-state solution $[(\partial/\partial t)P=0]$ W that satisfies the identity $LW=0$, and is given by

$$W(x)=N\exp\left[\frac{1}{D}\left[\frac{ax^2}{2}-\frac{bx^4}{4}\right]\right]\ , \tag{10}$$

in which N is a normalization constant. Eigenfunction-eigenvalue equations engendered by (6) are

$$LW\psi_n=-\lambda_n W\psi_n\ . \tag{11}$$

In the literature[11] on this problem, it is popular to define the eigenfunction-eigenvalue equations by

$$LW^{1/2}\phi_n=-\lambda_n W^{1/2}\phi_n\ . \tag{12}$$

Both choices yield identical eigenspectra. The choice in (12) produces equations for the ϕ_n's which have the form of the time-independent Schrödinger equation, whereas the choice in (11) produces equations for the ψ_n's which are of the general Sturm-Liouville variety. Thus the eigenfunctions for the two choices, while equivalent, differ in detailed structure and normalization. The Sturm-Liouville structure and the non-negativity of W make it immediately clear that the eigenspectrum is non-negative. The eigenfunction, unity, with eigenvalue zero, is nondegenerate. No other eigenfunction is obtainable in closed form.

Using Dirac notation, the eigenfunctions of (12) in the form $W^{1/2}\phi_n$ will be denoted by $|n\rangle$. The solution to (8) may be given a unique expansion in terms of the $|n\rangle$'s:

$$\bar{P}=\sum_{n=0}^{\infty}C_n(t)|n\rangle\ , \tag{13}$$

where the coefficients are given by

$$C_m(t)=\langle m|\bar{P}\rangle=\sum_{n=0}^{\infty}C_n(t)\langle m|n\rangle\ , \tag{14}$$

in which the Dirac notation means

$$\langle m|n\rangle=\int_{-\infty}^{\infty}dx\,\phi_m W^{1/2}W^{1/2}\phi_n=\delta_{mn}\ . \tag{15}$$

With this notation, Eq. (8) is transformed into the system of coupled equations for the coefficients:

$$\dot{C}_m=-\lambda_m C_m+\tfrac{1}{2}a_0^2\sum_{n,k}\left[\frac{\lambda_n-\lambda_k+\exp[-t(\lambda_n-\lambda_k)][\Omega\sin(\Omega t)-(\lambda_n-\lambda_k)\cos(\Omega t)]}{\Omega^2+(\lambda_n-\lambda_k)^2}\right]\left\langle m\left|\frac{\partial}{\partial x}\right|n\right\rangle\left\langle n\left|\frac{\partial}{\partial x}\right|k\right\rangle C_k\ . \tag{16}$$

The evenness of W in x implies that the $|n\rangle$ are also parity eigenstates. This has implications for the choices of m, n, and k that lead to nonzero matrix elements of the operator $\partial/\partial x$.

THE POWER SPECTRUM

To see stochastic resonance, it is necessary to obtain the power spectrum for $x(t)$. The power spectrum is the Fourier transform of the autocorrelation function for x. Suppose we solve (8), or equivalently (16), with the initial condition $\bar{P}(x,0)=\delta(x-x_0)$. The autocorrelation function $A(t)$ is

$$A(t)=\{\langle x(t)\rangle x_0\}\ , \tag{17}$$

in which $\langle\ \rangle$ denotes the average of x with respect to $\bar{P}(x,t)$, and $\{\ \}$ denotes the average over x_0 with respect to $W(x_0)$. This is valid if

$$W(x)=\lim_{t\to\infty}\bar{P}(x,t)\ . \tag{18}$$

Now, for Eq. (6) (no modulation), this limit is easily proven using the non-negativity of the eigenspectrum and the nondegeneracy of $\lambda_0=0$. It turns out that it is also true for Eq. (16) to lowest order in a_0 if a_0 is assumed to be small.

Let $\epsilon=a_0^2$ be a small parameter and write

$$C_m=C_m^{(0)}+\epsilon C_m^{(1)}+\epsilon^2 C_m^{(2)}+\cdots\ . \tag{19}$$

Putting this expansion into (16) produces an ϵ hierarchy of equations for the $C_m^{(i)}$'s. Clearly (zeroth order in ϵ),

$$C_m^{(0)}(t)=e^{-t\lambda_m}C_m^{(0)}(0)\ , \tag{20}$$

where $C_m^{(0)}(0)=W^{1/2}(x_0)\phi_m(x_0)$. To order ϵ,

$$\dot{C}_m^{(1)}=-\lambda_m C_m^{(1)}+\tfrac{1}{2}\sum_{n,k}\left[\frac{(\lambda_n-\lambda_k)e^{-t\lambda_k}+e^{-t\lambda_n}[\Omega\sin(\Omega t)-(\lambda_n-\lambda_k)\cos(\Omega t)]}{\Omega^2+(\lambda_n-\lambda_k)^2}\right]\left\langle m\left|\frac{\partial}{\partial x}\right|n\right\rangle\left\langle n\left|\frac{\partial}{\partial x}\right|k\right\rangle C_k^{(0)}(0)\ . \tag{21}$$

Therefore $A(t)$ can be expressed as

$$A(t)=\sum_{n=0}^{\infty}\{C_n(t)x_0\}\int_{-\infty}^{\infty}dx\,xW^{1/2}(x)\phi_n(x)\cong\sum_{n=0}^{\infty}[\{C_n^{(0)}(t)x_0\}+\epsilon\{C_n^{(1)}(t)x_0\}]\int_{-\infty}^{\infty}dx\,xW^{1/2}(x)\phi_n(x)\ . \tag{22}$$

The parity of the ϕ_n's implies that only odd n contribute to (22).

In (22), the $C_n^{(0)}$ terms produce contributions to the "noise" part of the power spectrum because they contain no a_0 terms and they provide the whole story in the absence of modulation. On the other hand, the $C_n^{(1)}$ terms contribute to the "signal' part of the power spectrum. The noise terms are Lorentzians [the Fourier transforms of the simple exponentials in (20)] given by

$$S_N(\omega)=\sum_{n=0}^{\infty}(\langle 2n+1|x_0\rangle)^2\frac{2\lambda_{2n+1}}{\omega^2+\lambda_{2n+1}^2}\ , \tag{23}$$

in which

$$\langle 2n+1|x_0\rangle\equiv\int_{-\infty}^{\infty}dx\ W^{1/2}(x)\phi_{2n+1}(x)x \tag{24}$$

and $S_N(\omega)$ is the noise power spectrum at frequency ω. The signal terms are mixtures of simple Lorentzians and shifted Lorentzians [the Fourier transforms of the exponentials and trigonometric functions in the solution to Eq. (21)] given by

$$\begin{aligned}S_S(\omega)=\tfrac{1}{2}a_0^2\sum_{m,n,k}&\left\langle m\left|\frac{\partial}{\partial x}\right|n\right\rangle\left\langle n\left|\frac{\partial}{\partial x}\right|k\right\rangle(\langle k|x_0\rangle)^2\frac{1}{\Omega^2+(\lambda_n-\lambda_k)^2}\frac{\lambda_n-\lambda_k}{\lambda_m-\lambda_k}\left[\frac{2\lambda_k}{\omega^2+\lambda_k^2}-\frac{2\lambda_m}{\omega^2+\lambda_m^2}\right]\\&+\frac{\Omega^2+(\lambda_n-\lambda_k)(\lambda_m-\lambda_n)}{\Omega^2+(\lambda_m-\lambda_n)^2}\frac{2\lambda_m}{\omega^2+\lambda_m^2}+\frac{-\Omega^2+(\lambda_n-\lambda_k)(\lambda_m-\lambda_n)}{\Omega^2+(\lambda_m-\lambda_n)^2}\left[\frac{\lambda_n}{(\omega+\Omega)^2+\lambda_n^2}+\frac{\lambda_n}{(\omega-\Omega)^2+\lambda_n^2}\right]\\&+\frac{\Omega(\lambda_m-\lambda_n)-\Omega(\lambda_n-\lambda_k)}{\Omega^2+(\lambda_m-\lambda_n)^2}\left[\frac{\omega+\Omega}{(\omega+\Omega)^2+\lambda_n^2}-\frac{\omega-\Omega}{(\omega-\Omega)^2+\lambda_n^2}\right]\ .\end{aligned} \tag{25}$$

The $n=0$ term in (25) is not Lorentzian since $\lambda_0=0$, but is instead $\pi\delta(\omega-\Omega)$. The solution to (21) needed for this result is easily obtained since the entire second term on the right-hand side of (21) is an inhomogeneous term. If we call it $I_m(t)$, then the solution to (21) has the form

$$C_m^{(1)}(t)=e^{-t\lambda_m}C_m^{(1)}(0)+\int_0^t ds\, e^{-(t-s)\lambda_m}I_m(s)\ , \qquad (26)$$

which is readily integrated. The $n=0$ term dominates the spectrum.

STOCHASTIC RESONANCE

These general expressions cannot be evaluated further without the explicit eigenfunctions and eigenvalues. Since these are not obtainable in closed form, several options may be pursued. Numerical procedures have been used[11] to gain insight into the structure of the eigenspectrum. Solvable models[12,13] of the double-well potentials have been used to gain additional insight regarding the eigenfunctions. As already mentioned in the Introduction, there is also the option of contracting the description[7–9] and thereby obtaining rate equations for the total probability in each of the two wells. In this paper we will use the first two options, and an approximation scheme.

The eigenspectrum possesses very interesting and important behavior. The first nonzero eigenvalue is λ_1, which corresponds with the eigenfunction of odd parity, ϕ_1. On numerical grounds,[11] it is known that for sufficiently high interwell barriers, or equivalently, for sufficiently small D, λ_1, approaches zero exponentially faster than all other nonzero eigenvalues, λ_m for $m\geq 2$. This means that for properly chosen parameter values, λ_1 is separated from the rest of the nonzero eigenspectrum. The Lorentzian contributions to (23) are dominated by the smallest eigenvalues. Upon first reflection, the λ_1 terms would appear to dominate. However, some care is required.

First of all, the λ_1 term clearly dominates the noise Lorentzians for sufficiently small ω. However, for stochastic resonance, the signal-to-noise ratio is computed for $\omega=\Omega$. It is possible that a different Lorentzian is bigger for $\omega=\Omega$ than is the λ_1 Lorentzian. In fact, two Lorentzians with different λ's and different coefficients, e.g.,

$$A_1\frac{\lambda_1}{\omega^2+\lambda_1^2}\quad\text{and}\quad A_3\frac{\lambda_3}{\omega^2+\lambda_3^2}\ ,$$

will cross at

$$\omega_0^2=(A_1\lambda_1\lambda_3^2-A_3\lambda_3\lambda_1^2)/(A_3\lambda_3-A_1\lambda_1)$$

provided that

$$\frac{\lambda_1}{\lambda_3}<\frac{A_3}{A_1}<\frac{\lambda_3}{\lambda_1}\ .$$

Thus, if we choose $\Omega>\omega_0$, the λ_3 Lorentzian will be bigger than the λ_1 Lorentzian at $\omega=\Omega$. If, however, $\Omega<\omega_0$, then the λ_1 Lorentzian is dominant.

Secondly, in the signal spectrum given by (25), there are the factors $\langle k|x_0\rangle$ which are nonzero only if k is odd because of parity. This means that n must be even because of the factor

$$\left\langle n\left|\frac{\partial}{\partial x}\right|k\right\rangle\ ,$$

again because of parity. Therefore the dominant term in (25) is the λ_0 δ function centered at $\omega=+\Omega$, and there is no such contribution for $n=1$, i.e., no such λ_1 term.

The consequences of these considerations are that the noise contributions to the spectrum are well approximated by

$$S_N(\omega)\cong(\langle 1|x_0\rangle)^2\frac{2\lambda_1}{\omega^2+\lambda_1^2} \qquad (27)$$

and the signal contributions are well approximated by

$$S_S(\omega)\cong-\tfrac{1}{2}a_0^2\left\langle 1\left|\frac{\partial}{\partial x}\right|0\right\rangle\left\langle 0\left|\frac{\partial}{\partial x}\right|1\right\rangle(\langle 1|x_0\rangle)^2 \times\frac{1}{\Omega^2}\pi\delta(\omega-\Omega) \qquad (28)$$

for $\omega\sim\Omega$ with $\Omega<\omega_0$ and $\Omega>\lambda_2$. The signal-to-noise ratio at resonance[9] ($\omega=\Omega$) is well approximated by

$$\frac{S}{N}\equiv\frac{S_S(\Omega)}{S_N(\Omega)}\cong-\tfrac{1}{2}a_0^2\left\langle 1\left|\frac{\partial}{\partial x}\right|0\right\rangle\left\langle 0\left|\frac{\partial}{\partial x}\right|1\right\rangle\frac{\pi}{\lambda_1}\ . \qquad (29)$$

In order to proceed further with this expression, quantitative results for the matrix elements, the inner product ratio, and the eigenvalues are needed.

APPROXIMATE TREATMENT

The expression in (29) is positive because the $\partial/\partial x$ matrix elements are negatives of each other on account of the skew-Hermitian character of $\partial/\partial x$ in the Schrödinger representation given by (12). In particular,

$$\left\langle 1\left|\frac{\partial}{\partial x}\right|0\right\rangle=\int_{-\infty}^{\infty}dx\,\phi_1(x)W^{1/2}(x)\frac{\partial}{\partial x}W^{1/2}(x)\phi_0(x)\ . \qquad (30)$$

Using (11) and (12) we see that $\psi_n=W^{1/2}\phi_n$ and can rewrite (11) as

$$D\frac{\partial}{\partial x}\exp[-f(x)/D]\frac{\partial}{\partial x}\exp[f(x)/D]\psi_n=-\lambda_n\psi_n\ . \qquad (31)$$

Risken[13] has shown that this differential equation is equivalent to the integral equation

$$\psi_n(x)=\exp[-f(x)/D]\left[\exp[f(\infty)/D]\psi_n(\infty)-\frac{\lambda_n}{D}\int_x^{\infty}dy\exp[f(y)/D]\int_y^{\infty}dz\,\psi_n(z)\right]\ , \qquad (32)$$

in which $x=\infty$ should be thought of in the sense of a limit value.

$f(x)$ is an even function of x, which implies that the ψ_n's are parity states. It is easily shown that $\psi_n(-x)$ is also a solution of (31) if $\psi_n(x)$ is. This must also be true for the integral equation (32), and it is, although the proof is much more subtle. Following Risken we obtain approximate solutions for $\psi_1(x)$ from (32) by iteration and imposition of odd parity. We obtain to lowest order

$$\begin{aligned}&\psi_1^{(0)}(x)\sim\exp[-f(x)/D] \quad \text{for } x\geq 0 ,\\ &\psi_1^{(0)}(x)=-\psi_1^{(0)}(|x|) \quad \text{for } x<0 ,\\ &\lambda_1^{(0)}=0 .\end{aligned} \tag{33}$$

To next order, we obtain

$$\begin{aligned}&\psi_1^{(1)}(x)\sim\exp[-f(x)/D]\left[1-\frac{\lambda_1^{(1)}}{D}\int_x^\infty dy\,\exp[f(y)/D]\int_y^\infty dz\,\exp[-f(z)/D]\right] \quad \text{for } x\geq 0 ,\\ &\psi_1^{(1)}(x)=-\psi_1^{(1)}(|x|) \quad \text{for } x<0 ,\\ &\lambda_1^{(1)}=D\left[\int_0^\infty dy\,\exp[f(y)/D]\int_y^\infty dz\,\exp[-f(z)/D]\right]^{-1} .\end{aligned} \tag{34}$$

With this approximation, and the fact that ψ_0 is known exactly: $\psi_0=N_0\exp[-f(x)/D]$, we finally obtain the matrix element

$$\begin{aligned}\left\langle 1\left|\frac{\partial}{\partial x}\right|0\right\rangle &=\int_{-\infty}^{\infty}dx\,\psi_1(x)\frac{\partial}{\partial x}\psi_0(x)\\ &=2\int_0^\infty dx\,\psi_1(x)\frac{\partial}{\partial x}\psi_0(x)\\ &=2N_0N_1\int_0^\infty dx\,\exp[-f(x)/D]\left[1-\frac{\lambda_1^{(1)}}{D}\int_x^\infty dy\,\exp[f(y)/D]\int_y^\infty dz\,\exp[-f(z)/D]\right]\\ &\qquad\times\left[-\frac{f'(x)}{D}\exp[-f(x)/D]\right]\\ &=2N_0N_1\int_0^\infty dx\tfrac{1}{2}\left[\frac{\partial}{\partial x}\exp[-2f(x)/D]\right]\left[1-\frac{\lambda_1^{(1)}}{D}\int_x^\infty dy\,\exp[f(y)/D]\int_y^\infty dz\,\exp[-f(z)/D]\right]\\ &=-N_0N_1\frac{\lambda_1^{(1)}}{D}\int_0^\infty dx\,\exp[-f(x)/D]\int_x^\infty dz\,\exp[-f(z)/D] .\end{aligned} \tag{35}$$

Putting this into (29) leads to an expression containing various normalization factors as well as the important factor

$$\frac{S}{N}\sim\left[\frac{\lambda_1^{(1)}}{D}\right]^2\frac{1}{\lambda_1} . \tag{36}$$

A variety of approximate approaches[11-13] (model potentials) show that

$$\lambda_1^{(1)}=\lambda_1\sim\exp[-\Delta U/D] .$$

This is the explanation of the exponential smallness of λ_1 for small D referred to in the preceding section. Thus we obtain the result quoted in Eq. (1).

CONCLUDING REMARKS

Since λ_1 determines the mean rate for hopping from one well to the other[14] as a result of the white noise g_w, it is now clear from the preceding discussion that *stochastic resonance* does *not* mean a resonance between the modulation frequency Ω and the mean hopping rate λ_1. The word "resonance" has to do with the choice of spectral frequency, $\omega=\Omega$, for the S/N enhancement as a function of D. Instead of the terminology *stochastic resonance,* it would be more appropriate to refer to *noise-induced signal-to-noise ratio enhancement,* a clumsier expression, but a much more accurate rendering.

In the ring laser experiment,[1] a signal-to-noise ratio enhancement was also observed at spectral frequency $\omega=2\Omega$. Its onset and offset were inside the profile for enhancement at $\omega=\Omega$. Clearly, the dynamics in (2) will generate odd harmonics of Ω, not an even harmonic. However, if the fourth cumulant correction to (8) is explored, it is seen to generate terms which contain 2Ω. Following these terms through to their contributions to the S/N ratio is greatly more involved than the procedure described above for the $\omega=\Omega$ enhancement, although it is straightforward in principle. Additional experimental results on the 2Ω harmonic will force completion of such an analysis and will provide a much more stringent test of the theory.

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APPENDIX

Define the operator L by

$$L=-\frac{\partial}{\partial x}(ax-bx^3)+D\frac{\partial^2}{\partial x^2} . \tag{A1}$$

Equation (7) may now be written as

$$\frac{\partial}{\partial t}P=LP-\frac{\partial}{\partial x}a_0\cos(\Omega t+\gamma)P . \tag{A2}$$

We wish to average this equation with respect to the uniformly distributed random phase γ. The difficulty we must confront is the noncommutativity of L and $\partial/\partial x$ (we let ∂_x denote $\partial/\partial x$ below).

$$P=\exp(tL)Q . \tag{A3}$$

Therefore

$$\frac{\partial}{\partial t}Q=-a_0\exp(-tL)\partial_x\exp(tL)\cos(\Omega t+\gamma)Q . \tag{A4}$$

This has a formal solution in terms of the time-ordered exponential:[10]

$$Q(t)=\overleftarrow{T}\exp\left[-\int_0^t ds\,\exp(-sL)\partial_x\exp(sL)a_0\times\cos(\Omega s+\phi)\right]Q(0) . \tag{A5}$$

This expression can be averaged with respect to γ in terms of ordered operator cumulants:

$$\langle Q(t)\rangle_\gamma=\overleftarrow{T}\exp\left[\int_0^t ds\sum_{m=1}^{\infty}C^{(m)}(s)\right]Q(0) , \tag{A6}$$

in which the first two cumulants are explicitly given by

$$\int_0^t ds\,C^{(1)}(s)=-a_0\int_0^t ds\,\exp(-sL)\partial_x\exp(sL)\langle\cos(\Omega t+\gamma)\rangle_\gamma =0 \tag{A7}$$

and

$$\begin{aligned}\int_0^t ds\,C^{(2)}(s)&=a_0^2\int_0^t ds\int_0^s ds'\exp(-sL)\partial_x\exp[(s-s')L]\partial_x\exp(s'L)\langle\cos(\Omega s+\gamma)\cos(\Omega s'+\gamma)\rangle_\gamma\\&=\tfrac{1}{2}a_0^2\int_0^t ds\int_0^s ds'\exp(-sL)\partial_x\exp[(s-s')L]\partial_x\exp(s'L)\cos\Omega(s-s') .\end{aligned} \tag{A8}$$

We omit all higher-order cumulants here because a_0 is small (i.e., $a_0 \ll a^{3/2}/b^{1/2}$). Note that this is a different reason for using only the second cumulant here than is used in Ref. 10 where a short correlation time is the reason invoked; here there is no correlation time for the γ fluctuations. Now, we can write, as a good approximation,

$$\frac{\partial}{\partial t}\langle Q\rangle_\gamma=C^{(2)}(t)\langle Q\rangle_\gamma . \tag{A9}$$

Writing $\bar{P}=\langle P\rangle_\gamma=\exp(tL)\langle Q\rangle_\gamma$ implies

$$\frac{\partial}{\partial t}\bar{P}=L\bar{P}+\exp(tL)C^{(2)}(t)\exp(-tL)\bar{P} , \tag{A10}$$

in which $C^{(2)}(t)$ is obtained from (A8) by differentiation with respect to t. The final result is precisely Eq. (8).

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