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Unstable evolution of pointwise trajectory solutions to chaotic maps

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Simple chaotic maps are used to illustrate the inherent instability of trajectory solutions to the Frobenius–Perron equation. This is demonstrated by the difference in the behavior of δ-function solutions and of extended densities. Extended densities evolve asymptotically and irreversibly into invariant measures on stationary attractors. Pointwise trajectories chaotically roam over these attractors forever. Periodic Gaussian distributions on the unit interval are used to provide insight. Viewing evolving densities as ensembles of unstable pointwise trajectories gives densities a stochastic interpretation. © *1995 American Institute of Physics.*

I. INTRODUCTION

A conservative classical dynamical system is said to exhibit chaos when its phase space trajectories show initially exponential sensitivity to small differences in initial conditions.[1] This property is made quantitative by showing that the largest Lyapunov exponent is positive.[2] The symplectic structure of classical, Hamiltonian dynamics implies that the sum of all of the Lyapunov exponents is zero,[3] which implies that at least one is positive, unless they all vanish as they do for integrable systems. Therefore, chaotic behavior is virtually generic for nonintegrable conservative Hamiltonian systems. The ability to accurately predict the future of a trajectory decays exponentially fast when chaos reigns. Nevertheless, chaos is not structureless. Representations of the dynamics, by cycle expansions[4] in terms of unstable periodic orbits in phase space, provide a rigid skeleton for chaotic dynamics, leading to short time predictability[5] and even the control of chaos.[5–7]

It has been argued[8,9] that chaos is the basis for a reconciliation between the time reversal invariance of conservative Hamiltonian dynamics and irreversible macroscopic observations. Chaos is said to generate intrinsic irreversibility, and it has even been suggested that there is an intrinsic connection between positive Lyapunov exponents and macroscopic transport coefficients.[10,11] This work has as its foundation in earlier work on Pollicott–Ruelle resonances in axiom-A systems.[12] The recent connections have been exhibited explicitly and constructively for invertible iterated maps.[8–10]

In closely related, independent research[13,14] growing out of the mathematical foundations of ergodic theory,[12] the idea of intrinsic irreversibility as a characteristic of chaotic dynamics exhibited by invertible maps has also risen. Particular emphasis is placed on the differences between the behavior of individual pointwise trajectories and of densities. While individual trajectories will indefinitely roam over the attracting phase space, densities may show monotonic, irreversible evolution toward asymptotic, invariant measures supported on the dynamical attractor for the map.

The orthodox physical interpretation of this difference is to view the evolving densities as ensembles of pointwise trajectories, in the spirit of the Boltzmann–Gibbs picture of the origin of irreversibility in Hamiltonian systems.[15] In this picture, Poincaré recurrences for individual trajectories argue against irreversibility, but because the member trajectories of an ensemble exhibit recurrences at vastly differing times, the ensemble as a whole does not recur. The cause of the initial density is a lack of knowledge about the precise initial state, a reason possibly having its origin in the partial ignorance of an outside observer.

Our own research in the general area of chaos has brought us into direct contact with these developments. Initially, we were concerned with the interplay of chaos and the dynamics of intrinsic, thermomolecular fluctuations in real physical systems.[16–19] We observed that the positivity of the largest Lyapunov exponent and the time evolution of the covariance matrix for the fluctuations were both determined by the evolving Jacobi matrix for the dynamics.[17] Specifically, chaotic dynamics in the deterministic description implies that initially small enough covariances will initially grow exponentially fast at a rate quantitatively determined in the same way as is the largest Lyapunov exponent. This connection with an experimentally measurable quantity, the growth rate of the covariances, changed our perspective on chaos. Rather than emphasize the exponential sensitivity to initial conditions felt by individual trajectories, we began to view chaos as manifested observationally in exponential growth of ensemble covariances.

In our recent study[20,21] of quantum signatures of classical chaos,[22] the ideas of phase space trajectories and Lyapunov exponents were carried over into quantum mechanics by means of Husimi–O'Connell–Wigner distributions.[23–26] These positive distributions have a direct classical phase space interpretation as ensembles. Using several model systems,[22,27] we were able to numerically follow the corresponding time evolutions of the Husimi–O'Connell–Wigner wave packets and of the corresponding classical ensembles. The initially exponential growth of the classical covariances mentioned above now manifested itself as initially exponential growth of the corresponding quantum covariances. This means that a measurement of the initial growth rate of initially small enough quantum covariances yields the classical Lyapunov exponent. (We note in passing, but refer the reader to the Refs. 20 and 21 for details, that this correspondence is quantitatively between the growth rate of the quantum covariances for the wave packet and the growth rate of the classical covariances as determined by the time evolution of the corresponding classical ensemble. This

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means that the growth rate is the *local expansion rate*, or "local" Lyapunov exponent,[28] as determined by the local initial conditions, and not the global, ergodic, phase space-averaged, Lyapunov exponent. Only when the local expansion rate is uniform over the attractor, is the local rate equal to the global Lyapunov exponent, as in, e.g. the Arnold Cat map.) Because Heisenberg's uncertainty principle forbids the initial quantum state from corresponding with a precise classical phase space point, we now find ourselves having to use initial classical states that are intrinsically densities and not singular, δ functions. The orthodox physical interpretation for these densities is as ensembles of precise classical points. Can we give these states a stochastic interpretation as well, i.e. as single extended states with a probabilistic description, more closely in keeping with their quantum origins?

Properties of the Frobenius–Perron operator are used in this paper. Its eigenvalues, which are related by resolvent operator methods to the Pollicott–Ruelle resonances,[12] play a central role. They not only determine the local Lyapunov exponents, as will be shown, but they also determine the asymptotic relaxation to the invariant measure.[12] For discrete maps, the asymptotic relaxation is always exponential (or products of exponentials and polynomials when the spectrum is degenerate)[9,12] because the eigenvalues are exponentials of strictly nonzero quantities, with the exception of the eigenvalue one, i.e. the exponential of zero, associated with the invariant measure.

In this paper, we address the preceding issues in a limited context that we believe is a paradigm for the key ideas. For this purpose, a few selected discrete maps are presented and analyzed. Nevertheless, the results obtained are applicable to a much wider range of dynamical systems that includes coupled ordinary differential equations. Such additional applications will be the subject of a sequel. The maps discussed here are the Bernoulli map;[8,9] the Tent map;[14] iterated function system (IFS) maps;[29] and the Baker map.[3,8,9,14,30,31] The first three cases are noninvertible, one-dimensional maps of the unit interval onto the unit interval, whereas the Baker map is an invertible, two-dimensional map of the unit square onto the unit square. While we review certain standard methods and results, e.g. the Frobenius–Perron operator equation[14] and the L_2 "shift-state" basis for these maps,[8,9] a technique not previously applied to these models is the centerpiece of our analysis. This technique utilizes the periodic Gaussian distribution on the unit interval.[32,33] It enables us to make physical statements about the dynamics that have otherwise remained unexplored. In particular, we show that chaos implies more than *exponential sensitivity to initial conditions;* it implies that *pointwise trajectory solutions are unstable.* The precise sense in which this is meant will become clear as we proceed, and will be exhibited as the noncommutativity of two limits. The stable solutions turn out to be evolving densities that may be viewed as ensembles of pointwise trajectories, or given an intrinsically stochastic interpretation, as single extended states with a probabilistic description.

Stochasticity arises in two ways, one of which correlates with the concept of the unstable manifold of chaotic dynamics, and the other of which correlates with the stable manifold. The Bernoulli map and the Tent map capture the essence of the first type of stochasticity, and the IFS maps capture the essence of the second type. Details are given in Secs. II–V. In Sec. VI, we show that the deterministic Baker map description (which is measure preserving and reversible, with one-dimensional unstable and stable manifolds), may be "contracted"[15] one way to yield the Bernoulli map,[3,30,31] and may be contracted in a complementary way to yield an IFS map.

This paper is arranged so that the details of the various maps are introduced in turn, along with the relevant methods. The presentation sequence is the Bernoulli map, the Tent map, the IFS maps, and the Baker map. The Frobenius–Perron equation, the shift-state basis, and the periodic Gaussian distributions are introduced along with the Bernoulli map. The Bernoulli polynomial eigenstates[8,9] for the Bernoulli map, Frobenius–Perron equation are presented with the Bernoulli map, but the details are relegated to Appendix A. Similar eigenstates for the Tent map are also discussed in Appendix B. Following the details regarding the maps and the results for the two contractions of the description for the Baker map, we will summarize our interpretation of these results in Sec. VII of the paper. There, mention is made of the Quadratic map, and its eigenvalues and eigenfunctions are presented in Appendix C. This enables us to show how to generalize the results obtained for the Bernoulli and Tent maps to generic one-dimensional maps.

II. THE BERNOULLI MAP

The Bernoulli map is the simplest chaotic map.[8,9] It is defined by

$$x_{n+1}=D_{\mathrm{B}}(x_n), \tag{1}$$

in which

$$D_{\mathrm{B}}(x)=2x \ \mathrm{MOD}\ 1, \tag{2}$$

for $x\in[0,1)$, or by

$$D_{\mathrm{B}}(x)=\begin{cases}2x,\\ 2x-1,\end{cases} \tag{3}$$

wherein the upper expression holds for $0\leqslant x<\frac{1}{2}$, and the lower expression holds for $\frac{1}{2}\leqslant x<1$. It maps the unit interval onto the unit interval, and every point, x, has two inverse images: $x/2$ and $x/2+\frac{1}{2}$. One way to view this map's behavior is to simply start with an initial x, x_0 say, and then iterate the map, thereby producing a trajectory of points. It is well known and easy to show that such trajectories are chaotic, exhibiting exponential sensitivity to initial x_0 variations. The Lyapunov exponent, λ_{L}, is ln 2.

Another way to view this map's behavior is through the Frobenius–Perron operator equation[3,8,14] that describes the evolution of a density on the unit interval. In this case, the Frobenius–Perron equation is

$$P_{\mathrm{B}}f(x)=\int_0^1 dy\ \ \delta[x-D_{\mathrm{B}}(y)]f(y), \tag{4}$$

in which $P_{\rm B}$ denotes the Frobenius–Perron operator for the Bernoulli map. Let $\{D_{\rm B}^{-1}(x)\}$ denote the set of inverse images of the point x. Well-known properties of the Dirac δ function imply

$$P_{\rm B}f(x)=\sum_{\{D_{\rm B}^{-1}(x)\}}\int_0^1 dy\ \frac{\delta[y-D_{\rm B}^{-1}(x)]}{|D'_{\rm B}[D_{\rm B}^{-1}(x)]|}\ f(y)$$
$$=\sum_{\{D_{\rm B}^{-1}(x)\}}\frac{f[D_{\rm B}^{-1}(x)]}{|D'_{\rm B}[D_{\rm B}^{-1}(x)]|}$$
$$=\frac{1}{2}f\left(\frac{x}{2}\right)+\frac{1}{2}f\left(\frac{x}{2}+\frac{1}{2}\right) \tag{5}$$

in which $D'_{\rm B}$ denotes the derivative of $D_{\rm B}$, and equals 2 for all x.

The trajectory solutions may be represented by Dirac δ-function solutions of the Frobenius–Perron equation. Define $f_\delta^{(n)}(x)$ by

$$f_\delta^{(n)}(x)=\delta(x-x_n), \tag{6}$$

where x_n is the nth iterate of $D_{\rm B}$ starting from x_0. Clearly

$$P_{\rm B}f_\delta^{(n)}(x)=\int_0^1 dy\ \ \delta[x-D_{\rm B}(y)]\,\delta(y-x_n)$$
$$=\delta[x-D_{\rm B}(x_n)]=\delta(x-x_{n+1})$$
$$=f_\delta^{(n+1)}(x), \tag{7}$$

wherein we have used Eqs. (1) and (6).

One should note that the MOD function in Eq. (1) makes the Bernoulli map nonlinear, whereas the Frobenius–Perron equation is a linear integral equation. Clearly, it is wrong to simply say that a linear equation cannot exhibit chaos because here we see that the linear Frobenius–Perron equation has special, Dirac δ-function solutions that follow exactly the chaotic trajectories for the nonlinear Bernoulli map. This situation is paralleled by Dirac δ-function solutions to the linear, partial differential, Liouville equation of Hamiltonian classical mechanics that follow exactly the chaotic phase space trajectories generated by Hamilton's equations, generally a system of coupled, nonlinear, ordinary differential equations.[17] For this reason, the results we describe below for simple maps have direct analogs in much more complicated, continuous time, dynamics.

Several kinds of basis functions are of interest when the Bernoulli map is studied. These include[8,9] shift states, modified Legendre polynomials, and Bernoulli polynomials. The shift states, $e_{n,k}(x)$, are defined[8,9] by

$$e_{n,k}(x)=\ \exp[i2\pi2^n(2k+1)x], \tag{8}$$

where n is any non-negative integer and k is any integer. These states form an L_2 basis satisfying orthonormality:

$$\int_0^1 dx\ e_{n,k}(x)^*e_{n',k'}(x)=\delta_{nn'}\ \delta_{kk'}, \tag{9}$$

and completeness:

$$1+\sum_{nk}\ e_{n,k}(x_0)^*e_{n,k}(x)=\sum_{j=-\infty}^{\infty}\ \delta(x-x_0+j). \tag{10}$$

It is important to note for later reference that completeness is given in terms of the periodic δ function, even though, for x and x_0 in the unit interval, only the $j=0$ term is nonzero. The shift states get their name from the property

$$P_{\rm B}e_{n,k}(x)=\frac{1}{2}\ \exp\left(i2\pi2^n(2k+1)\ \frac{x}{2}\right)$$
$$+\frac{1}{2}\ \exp\left[i2\pi2^n(2k+1)\left(\frac{x}{2}+\frac{1}{2}\right)\right]$$
$$=\frac{1}{2}\ \{1+\ \exp[i2\pi2^{n-1}(2k+1)]\}e_{n-1,k}(x)$$
$$=e_{n-1,k}(x), \tag{11}$$

for $n\geqslant1$, and equals 0 for $n=0$.

Together, Eqs. (10) and (11) provide an instructive variation on Eq. (7):

$$P_{\rm B}\sum_{j=-\infty}^{\infty}\ \delta(x-x_0+j)$$
$$=P_{\rm B}\left(1+\sum_{n=0}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n,k}(x_0)^*e_{n,k}(x)\right)$$
$$=1+\sum_{n=0}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n,k}(x_0)^*P_{\rm B}e_{n,k}(x)$$
$$=1+\sum_{n=1}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n,k}(x_0)^*e_{n-1,k}(x)$$
$$=1+\sum_{n=0}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n+1,k}(x_0)^*e_{n,k}(x)$$
$$=1+\sum_{n=0}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n,k}(2x_0)^*e_{n,k}(x)$$
$$=1+\sum_{n=0}^{\infty}\ \sum_{k=-\infty}^{\infty}\ e_{n,k}[D_{\rm B}(x_0)]^*e_{n,k}(x)$$
$$=\sum_{j=-\infty}^{\infty}\ \delta[x-D_{\rm B}(x_0)+j]. \tag{12}$$

We will return to this representation below.

The Bernoulli polynomials are the eigenstates of $P_{\rm B}$. Their definition and properties are given in Appendix A. They do not yield an L_2 basis for [0,1) nor are they orthogonal. Nevertheless, the fact that they are eigenfunctions, as well as the magnitudes of their eigenvalues, has prompted much interest.[8,9] From the generating function [see Eq. (A14)]

$$\frac{t\ \exp[xt]}{\exp[t]-1}=\sum_{m=0}^{\infty}\ B_m(x)\ \frac{t^m}{m!}, \tag{13}$$

it follows that

$$P_{\mathrm{B}} \frac{t \exp[xt]}{\exp[t]-1} = \frac{1}{2} \frac{t \exp[(x/2)t]}{\exp[t]-1} + \frac{1}{2} \frac{t \exp[(x/2+\frac{1}{2})t]}{\exp[t]-1}$$
$$= \frac{(t/2)\exp[x(t/2)]}{\exp[t/2]-1} = \sum_{m=0}^{\infty} B_m(x) \frac{(t/2)^m}{m!}. \tag{14}$$

Therefore

$$\sum_{m=0}^{\infty} P_{\mathrm{B}} B_m(x) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \frac{1}{2^m} B_m(x) \frac{t^m}{m!}, \tag{15}$$

which implies

$$P_{\mathrm{B}} B_m(x) = \frac{1}{2^m} B_m(x). \tag{16}$$

Thus, the mth eigenvalue can be expressed in terms of the Lyapunov exponent for the Bernoulli map:

$$\frac{1}{2^m} = \exp[-m\lambda_{\mathrm{L}}]. \tag{17}$$

This simple connection between the eigenvalues and the Lyapunov exponent, which is equivalent to a connection between the Pollicott–Ruelle resonances[12] and the Lyapunov exponent, cannot be expected to remain so simple for more complicated maps. This issue is explored in the following sections. Iteration of P_{B} causes the Bernoulli polynomial eigenstates to decay, except for the $m=0$ case, $B_0(x)=1$, which is the invariant measure.

The lack of orthogonality and the failure to form an L_2 basis has been the focus of earlier research,[8,9] wherein it has been shown how to enlarge the Hilbert space in such a way as to rectify these difficulties. Here, we will take a quite different approach. In this approach, the initial time evolution may be expressed in terms of the natural, shift state basis, if a Hilbert space representation is desired. However, we will be directly concerned with periodic Gaussian distributions instead.

III. PERIODIC GAUSSIAN DISTRIBUTIONS

Gaussian distributions on infinite domains are extremely useful in many contexts. To achieve comparable utility on finite intervals requires the use of the periodic Gaussian distribution,[32,33] defined by

$$f_\sigma(x;x_0) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x-x_0+n)^2}{2\sigma^2}\right). \tag{18}$$

This expression has a shift state expansion [cf. Eq. (10)]

$$f_\sigma(x;x_0) = 1 + \sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} e_{n,k}[(x_0)^* e_{n,k}(x)]$$
$$\times \exp[-2\pi^2 2^{2n}(2k+1)^2\sigma^2], \tag{19}$$

which is proved as follows:

$$\int_0^1 dx \exp[-i2\pi mx] \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x-x_0+n)^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \int_0^1 dx \exp[-i2\pi mx]\exp\left(-\frac{(x-x_0+n)^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \int_n^{1+n} dx \exp[-i2\pi m(x-n)]\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \int_n^{1+n} dx \exp[-i2\pi mx]\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{(2\pi\sigma^2)}} \int_{-\infty}^{\infty} dx \exp[-i2\pi mx]\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right) = \exp[-i2\pi mx_0]\exp[-2\pi^2 m^2\sigma^2]. \tag{20}$$

Letting $m \to 2^n(2k+1)$ completes the proof. For $m=0$, this proves that $f_\sigma(x;x_0)$ is normalized to unity on the interval [0,1).

Now consider the action of P_{B} on $f_\sigma(x;x_0)$:

$$P_{\mathrm{B}} f_\sigma(x;x_0) = \frac{1}{2} f_\sigma\left(\frac{x}{2};x_0\right) + \frac{1}{2} f_\sigma\left(\frac{x}{2}+\frac{1}{2};x_0\right)$$
$$= \frac{1}{2} \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x/2-x_0+n)^2}{2\sigma^2}\right) + \exp\left(-\frac{(x/2+\frac{1}{2}-x_0+n)^2}{2\sigma^2}\right)\right]$$
$$= \frac{1}{\sqrt{[2\pi(2\sigma)^2]}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-2x_0+2n)^2}{2(2\sigma)^2}\right) + \exp\left(-\frac{(x-2x_0+2n+1)^2}{2(2\sigma)^2}\right)\right]$$

$$= \frac{1}{\sqrt{[2\pi(2\sigma)^2]}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x-2x_0+n)^2}{2(2\sigma)^2} \right) = \frac{1}{\sqrt{[2\pi(2\sigma)^2]}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(x-D_B(x_0)+n)^2}{2(2\sigma)^2} \right)$$
$$= f_{2\sigma}[x;D_B(x_0)]. \tag{21}$$

We see that the periodic Gaussian distribution propagates as a periodic Gaussian, but that its standard deviation, σ, has doubled. In our work on quantum signatures of classical chaos,[20,21] it was the growth rate of the standard deviations for initially Gaussian wave packets, and for their corresponding classical ensembles, that manifested the chaos, and that was determined by the ''local'' Lyapunov exponent.[28] For the Bernoulli map, the Lyapunov exponent, $\lambda_L = \ln 2$, is a global property reflecting an expansion rate that is uniform over the entire interval $[0,1)$. Thus, our earlier work[17,18] would predict a standard deviation growth rate given by $\exp[n\lambda_L]$ after n iterations of P_B. This is precisely what we see in Eq. (21).

Compare Eq. (19) with Eq. (10). It is clear that

$$\underset{\sigma\to 0}{\text{limit}}\, f_\sigma(x;x_0) = \sum_{j=-\infty}^{\infty} \delta(x-x_0+j). \tag{22}$$

Denote the n-fold iteration of P_B by P_B^{0n}, on the n-fold iteration of D_B by D_B^{0n}. We may now write

$$\underset{n\to\infty}{\text{limit}}\, P_B^{0n} f_\sigma(x;x_0) = 1 \tag{23}$$

and

$$\underset{n\to\infty}{\text{limit}}\, P_B^{0n} \sum_{j=-\infty}^{\infty} \delta(x-x_0+j)$$
$$= \underset{n\to\infty}{\text{limit}} \sum_{j=-\infty}^{\infty} \delta[x-D_B^{0n}(x_0)+j]. \tag{24}$$

The first limit yields the P_B-invariant equilibrium measure, 1. This results from the divergence of the standard deviation. The second limit once again illustrates the preservation of pointwise trajectories by Dirac δ-function solutions. It is now manifest that the two limits, $\sigma\to 0$ and $n\to\infty$, do not commute.

This result is our first important point of emphasis. Here $f_\sigma(x;x_0)$, for very small but nonzero σ, is close to $\sum_{j=-\infty}^{\infty}\delta(x-x_0+j)$ in a function space metrical sense. It is an extended, functional variation of the δ function. With respect to the action of P_B^{0n}, for large n, the δ-function solutions are unstable in the sense that $f_\sigma(x;x_0)$ variations for $\sigma\neq 0$, no matter how small σ, will irreversibly approach 1, the uniform density. There is a clear dichotomy in the asymptotic n behavior between the δ-function trajectories and the densities with nonzero standard deviations. These non-δ-function densities are ensembles of pointwise trajectories.

By using $f_\sigma(x;x_0)$, we capture the irreversible behavior inherent in the Bernoulli map, but not observed in individual trajectories, and even observe the contribution of the Lyapunov exponent to the rate of relaxation to equilibrium. In addition, there is a connection between $f_\sigma(x;x_0)$ and the $B_m(x)$'s that is illuminating and for which we make the following digression, as a conclusion to this section.

It is known[8,34] that a periodic function on $[0,1)$ cannot be expanded in terms of the $B_m(x)$'s. We will see why below. From the calculus of finite differences,[34] we find that a function, $f(x)$, has a unique expansion in terms of the $B_m(x)$'s:

$$f(x) = \sum_{m=0}^{\infty} c_m B_m(x), \tag{25}$$

in which the coefficients, c_m, are given by

$$c_m = \frac{1}{m!}[f^{(m-1)}(1) - f^{(m-1)}(0)], \tag{26}$$

where $f^{(k)}$ denotes the kth derivative of f. From Eq. (A19) we see that

$$\int_0^1 dx\, B_m(x) = \delta_{m0}, \tag{27}$$

which implies

$$c_0 = \int_0^1 dx\, f(x). \tag{28}$$

Using Eq. (A17), we also see that

$$\int_0^1 dx \frac{d}{dx} f(x) = \int_0^1 dx \sum_{m=0}^{\infty} c_m \frac{d}{dx} B_m(x)$$
$$= \sum_{m=1}^{\infty} c_m m \int_0^1 dx\, B_{m-1}(x) = c_1. \tag{29}$$

Similarly,

$$\frac{d^2}{dx^2} f(x) = \sum_{m=2}^{\infty} c_m m(m-1) B_{m-2}(x), \tag{30}$$

so that upon integration, using Eq. (27) gives

$$c_2 = \tfrac{1}{2}[f^{(1)}(1) - f^{(1)}(0)]. \tag{31}$$

Repeating this process n times yields

$$f^{(n)}(x) = \sum_{m=n}^{\infty} c_m \frac{m!}{(m-n)!} B_{m-n}(x) \tag{32}$$

and

$$c_n = \frac{1}{n!}[f^{(n-1)}(1) - f^{(n-1)}(0)], \tag{33}$$

which concludes the proof. Obviously, for periodic $f(x), f(1) = f(0)$, and $f^{(m)}(1) = f^{(m)}(0)$ for all m, which implies $c_m = 0$ for all $m \geqslant 1$. This verifies the claim that a periodic $f(x)$ cannot be expanded in terms of the $B_m(x)$'s. Thus, $f_\sigma(x;x_0)$ cannot be expanded in this way.

From a physical perspective, what is important is how good an approximation to $f_\sigma(x;x_0)$ can be achieved in terms of the $B_m(x)$'s. A very accurate expansion in the $B_m(x)$'s is in fact possible. For $\sigma \ll 1$, three terms in Eq. (18) overwhelmingly dominate. They are the terms with $n=-1,\ 0,$ and $+1$. For x_0 more than a few σ's away from either 0 or 1, the $n=0$ term is the only significant contribution, but for x_0 within a few σ's of 1 or 0, the $n=-1$ or $n=+1$ terms may also become non-negligible. Thus, a very highly accurate approximation for $\sigma \ll 1$ is

$$f_\sigma(x;x_0) \approx \sum_{n=-1}^{+1} \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left(- \frac{(x-x_0+n)^2}{2\sigma^2} \right), \tag{34}$$

which is no longer periodic in x. Even this function gives us trouble when the expansion coefficients in Eq. (33) are computed, because they diverge as their index increases. However, a polynomial approximation to the three Gaussians in Eq. (34) of order $2N$ is possible and is very good provided $N > e/2\sigma^2$. For small σ, N is very large, which reflects the latent divergence of the expansion. This approximation does have a unique expansion in the $B_m(x)$'s:

$$\begin{aligned} f_\sigma^{(N)}(x;x_0) &\approx \sum_{n=-1}^{+1} \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{m=0}^{N} \frac{(-1)^m}{m!} \left(\frac{(x-x_0+n)^2}{2\sigma^2} \right)^m \\ &\approx \sum_{m=0}^{\infty} c_m^{(\sigma)} B_m(x), \end{aligned} \tag{35}$$

and we see that

$$P_{\mathrm{B}}^{0n} \sum_{m=0}^{\infty} c_m^{(\sigma)} B_m(x) = \sum_{m=0}^{\infty} c_m^{(\sigma)} \frac{1}{2^{nm}} B_m(x), \tag{36}$$

which approaches 1 as $n \to \infty$ since $c_0^{(\sigma)}$ is a very accurate approximation to 1. This exhibits the relaxation of $f_\sigma(x;x_0)$ to equilibrium with decay rates manifested in $\exp[-m\lambda_{\mathrm{L}}]$, as noted before. Even though very many eigenvalues are involved in determining the initial time evolution and the asymptotic relaxation, they all depend on the single parameter, $\lambda_{\mathrm{L}} = \ln 2$.

IV. THE TENT MAP

The Tent map[14] is defined for $x \in [0,1]$ by

$$x_{n+1} = D_{\mathrm{T}}(x_n), \tag{37}$$

where

$$D_{\mathrm{T}}(x) = \begin{cases} 2x, \\ 2-2x, \end{cases} \tag{38}$$

wherein the upper expression holds for $0 \leqslant x < \frac{1}{2}$ and the lower expression holds for $\frac{1}{2} \leqslant x \leqslant 1$. It is straightforward to show from equations paralleling Eqs. (4) and (5) that the Frobenius–Perron equation for the Tent map is

$$P_{\mathrm{T}} f(x) = \frac{1}{2} f\left(\frac{x}{2} \right) + \frac{1}{2} f\left(1 - \frac{x}{2} \right). \tag{39}$$

We analyze this case in order to make several observations by way of comparison with the results for the Bernoulli map. Once again the shift states prove to be very useful. Moreover, the periodic Gaussian distributions again enable us to exhibit the noncommutativity of the limit $\sigma \to 0$ and $n \to \infty$. There are also eigenstates for the Tent map (see Appendix B) that are very closely related to the Bernoulli polynomials for the Bernoulli map. These eigenstates also permit expansions such as in Eq. (25). We return to this point below.

The action of P_{T} on the shift states is given by

$$\begin{aligned} P_{\mathrm{T}} e_{n,k}(x) &= \frac{1}{2} \exp\left(i2\pi 2^n (2k+1) \frac{x}{2} \right) \\ &\quad + \frac{1}{2} \exp\left[i2\pi 2^n (2k+1) \left(1 - \frac{x}{2} \right) \right] \\ &= \frac{1}{2} e_{n-1,k}(x) + \frac{1}{2} e_{n-1,k}(x)^*. \end{aligned} \tag{40}$$

This implies the identities for $n \geqslant 0$,

$$P_{\mathrm{T}} \cos[2\pi 2^n (2k+1)x] = \cos[2\pi 2^{n-1}(2k+1)x], \tag{41}$$

$$P_{\mathrm{T}} \cos[\pi(2k+1)x] = 0, \tag{42}$$

$$P_{\mathrm{T}} \sin[2\pi 2^n (2k+1)x] = 0. \tag{43}$$

Equations (42) and (43) are analogs of the "collapsing" states found earlier[8,9] for the Bernoulli map. The shift states are an orthonormal L_2 basis for the Tent map.

The action of P_{T} on $f_\sigma(x;x_0)$ is a bit more complicated than the action of P_{B}. Define $f_\sigma^{(p)}(x;x_0)$ by

$$f_\sigma^{(p)}(x;x_0) = \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \exp\left(- \frac{(x-x_0+pn)^2}{2\sigma^2} \right). \tag{44}$$

Our earlier $f_\sigma(x;x_0)$ is simply $f_\sigma^{(1)}(x;x_0)$. Now we find

$$\begin{aligned} P_{\mathrm{T}} f_\sigma^{(1)}(x;x_0) &= \frac{1}{2} \frac{1}{\sqrt{(2\pi\sigma^2)}} \sum_{n=-\infty}^{\infty} \left[\exp\left(- \frac{(x/2-x_0+n)^2}{2\sigma^2} \right) \right. \\ &\quad \left. + \exp\left(- \frac{(1-x/2-x_0+n)^2}{2\sigma^2} \right) \right] \\ &= \frac{1}{\sqrt{[2\pi(2\sigma)^2]}} \sum_{n=-\infty}^{\infty} \left[\exp\left(- \frac{(x-2x_0+2n)^2}{2(2\sigma)^2} \right) \right. \\ &\quad \left. + \exp\left(- \frac{[x-(2-2x_0)+2n]^2}{2(2\sigma)^2} \right) \right] \\ &= f_{2\sigma}^{(2)}(x;2x_0) + f_{2\sigma}^{(2)}(x;2-2x_0), \end{aligned} \tag{45}$$

wherein the interchange of $-n$ and n was made in the second summand between the first and second equalities. That this action preserves the normalization is seen as follows:

$$\begin{aligned} &\int_0^1 dx\ P_{\mathrm{T}} f_\sigma^{(1)}(x;x_0) \\ &\quad = \int_0^1 dx\ f_{2\sigma}^{(2)}(x;2x_0) + \int_0^1 dx\ f_{2\sigma}^{(2)}(x;2-2x_0) \end{aligned} \tag{46}$$

and

$$\int_0^1 dx\ f_{2\sigma}^{(2)}(x;2x_0)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\sum_{n=-\infty}^{\infty}\int_0^1 dx\ \exp\left(-\frac{(x-2x_0+2n)^2}{2(2\sigma)^2}\right)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\sum_{n=-\infty}^{\infty}\int_{2n}^{1+2n} dx\ \exp\left(-\frac{(x-2x_0)^2}{2(2\sigma)^2}\right) \tag{47}$$

and

$$\int_0^1 dx\ f_{2\sigma}^{(2)}(x;2-2x_0)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\sum_{n=-\infty}^{\infty}\int_0^1 dx$$
$$\times\exp\left(-\frac{[x-(2-2x_0)+2n]^2}{2(2\sigma)^2}\right)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\sum_{n=-\infty}^{\infty}\int_{2(n-1)}^{1+2(n-1)} dx\ \exp\left(-\frac{(x+2x_0)^2}{2(2\sigma)^2}\right)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\sum_{n=-\infty}^{\infty}\int_{2(n+1)-1}^{2(n+1)} dx$$
$$\times\exp\left(-\frac{(x-2x_0)^2}{2(2\sigma)^2}\right), \tag{48}$$

wherein the last equality follows from the interchange of $-x$ and x, and the interchange of $-n$ and n. Adding together the results of Eqs. (47) and (48) yields

$$\int_0^1 dx\ P_{\mathrm{T}} f_\sigma^{(1)}(x;x_0)$$
$$=\frac{1}{\sqrt{[2\pi(2\sigma)^2]}}\int_{-\infty}^{\infty} dx\ \exp\left(-\frac{(x-2x_0)^2}{2(2\sigma)^2}\right)=1, \tag{49}$$

which was to be proved.

If we look carefully at the right-hand side of Eq. (45), we see that for given x_0, only one of the two expressions has a large peak for $x\in[0,1]$. For $0\leqslant x_0<\frac{1}{2}$, it is the first expression, whereas for $\frac{1}{2}\leqslant x_0\leqslant 1$, it is the second. It is also clear that the limit in Eq. (22) still holds, so that for $\sigma=0$ we have a Dirac δ-function solution that follows pointwise trajectories emanating from x_0 by iteration of D_{T}. However, for $\sigma\neq 0$, iteration of P_{T} repeatedly doubles the standard deviation σ, as is clear from Eq. (45), and as was true for P_{B}. Thus, in the limit $n\rightarrow\infty$, we obtain the analog of Eq. (23),

$$\operatorname*{limit}_{n\rightarrow\infty} P_{\mathrm{T}}^{0n} f_\sigma^{(1)}(x;x_0)=1. \tag{50}$$

Once again, the limits $\sigma\rightarrow 0$ and $n\rightarrow\infty$ do not commute. Trajectory solutions to the Frobenius–Perron equation for the Tent map are unstable with respect to functional variations having nonvanishing standard deviations.

Since the Lyapunov exponent for the Tent map is also $\lambda_{\mathrm{L}}=\ln 2$, we see from Eq. (45) that the rate by which the limit in Eq. (50) is achieved is determined again by λ_{L}. This observation is not quite as easy to make as it was for the Bernoulli map because of the difference in the right-hand side of Eqs. (21) and (45). From Eq. (21), we readily obtain

$$P_{\mathrm{B}}^{0n} f_\sigma(x;x_0)=f_{2^n\sigma}[x;D_{\mathrm{B}}^{0n}(x_0)], \tag{51}$$

whereas from Eq. (45) we obtain, for example,

$$P_{\mathrm{T}}^{02} f_\sigma^{(1)}(x;x_0)=f_{4\sigma}^{(4)}(x;4x_0)+f_{4\sigma}^{(4)}(x;2-4x_0)$$
$$+f_{4\sigma}^{(4)}[x;2-(4-4x_0)]+f_{4\sigma}^{(4)}(x;4-4x_0). \tag{52}$$

For a given x_0, only one of the four expressions on the right-hand side has a large peak for $x\in[0,1]$. For $0\leqslant x_0<\frac{1}{4}$, $\frac{1}{4}\leqslant x_0<\frac{1}{2}$, $\frac{1}{2}\leqslant x_0<\frac{3}{4}$, and $\frac{3}{4}\leqslant x_0\leqslant 1$, it is the first, second, third, and fourth expression, respectively, that is peaked. Near any boundary point, e.g. $\frac{1}{4}$, the two neighboring expressions will be nearly the same. Nevertheless, iteration of P_{T} will clearly repeatedly double the standard deviation, σ, while the dominant peak will dance around among increasing numbers of disjoint expressions that generalize Eq. (52). It is this doubling of the standard deviation that exhibits the irreversible effect of a positive Lyapunov exponent.

For the Bernoulli map, we were able to exhibit the connection with the Lyapunov exponent through the behavior of the Bernoulli polynomial eigenstates and the $f_\sigma^{(N)}$ expansion, Eq. (35). The Frobenius–Perron equation for the Tent map also has eigenstates (see Appendix B), which satisfy

$$P_{\mathrm{T}} E_{2m}(x)=\frac{1}{2^{2m}}\, ET_{2m}(x),\quad \text{for}\ \ m=0,1,2,\ldots, \tag{53}$$

$$P_{\mathrm{T}} E_{2m+1}(x)=0,\quad \text{for}\ \ m=0,1,2,\ldots\ . \tag{54}$$

By parallelling the argument used in Eq. (25)–Eq. (36), the analog to Eq. (36) is obtained,

$$P_{\mathrm{T}}^{0n}\sum_{m=0}^{\infty} d_{2m}^{(\sigma)} ET_{2m}(x)=\sum_{m=0}^{\infty} d_{2m}^{(\sigma)}\,\frac{1}{2^{n2m}}\, ET_{2m}(x). \tag{55}$$

This time we see the relaxation of $f_\sigma^{(1)}(x;x_0)$ to equilibrium with decay rates manifested in $\exp[-2m\lambda_{\mathrm{L}}]$ in parallel with the situation for the Bernoulli map [see Eq. (17) and Appendix B for details]. Nevertheless, notice that there is an extra factor of 2, showing that a bit more complicated connection between the eigenvalues and the Lyapunov exponent has arisen. This may be expressed by saying that the first nontrivial Pollicott–Ruelle resonance is $\frac{1}{4}$ in this case, rather than $\frac{1}{2}$, as in the Bernoulli map case. However, all eigenvalues are multiples of $2\lambda_{\mathrm{L}}=2\ln 2$, a single parameter.

V. ITERATED FUNCTION SYSTEMS (IFS)

The Bernoulli map generates one type of stochastic process when the evolving density, $f_\sigma(x;x_0)$, is interpreted as a probability distribution. A different type of one-dimensional stochastic process is the iterated function system (IFS).[29,35] An especially instructive example[29] is the IFS constructed from the two maps,

$$M_1(x)=x/3, \tag{56}$$

$$M_2(x)=x/3+\tfrac{2}{3}, \tag{57}$$

by the iteration rule that selects one or the other map, each with probability $\frac{1}{2}$. For $x\in[0,1]$, the attractor for just M_1 alone is the point at $x=0$, while the attractor for just M_2 alone is the point at $x=1$. The attractor for the IFS created by stochastically iterating together the two maps is the Cantor set on $[0,1]$. It is elementary to prove this if one uses the ternary representation[36] for the reals on $[0,1]$:

$$x=\sum_{n=1}^{\infty}\frac{t_n}{3^n}, \tag{58}$$

in which each t_n has three possible values: 0, 1, and 2. We may equally well express each x by the ternary "decimal" expansion: $0.t_1t_2t_3\cdots$. The actions of M_1 and M_2 are succinctly expressed by

$$M_1(x)=0.0t_1t_2t_3\cdots, \tag{59}$$

$$M_2(x)=0.2t_1t_2t_3\cdots. \tag{60}$$

It is obvious that

$$\underset{n\to\infty}{\mathrm{limit}}\; M_1^{0n}(x)=0.000\cdots=0, \tag{61}$$

$$\underset{n\to\infty}{\mathrm{limit}}\; M_2^{0n}(x)=0.222\cdots=1, \tag{62}$$

as asserted above. It is also obvious that the attractor for the IFS is all possible ternary decimal expansions containing only 0's and 2's. By excluding all ternary decimal expansions containing 1's, we are geometrically excluding all of the "middle third" intervals in $[0,1]$, leaving an uncountable Cantor set.[29,36] Because this attractor set is isomorphic to the set of binary decimal expansions (see below) for $x\in[0,1]$, as is seen by identifying ternary 2's with binary 1's, it is as uncountable as the reals. Yet, it is nowhere dense,[36] has measure zero,[36] and fractal dimension $\ln 2/\ln 3$.[37]

Consider a different IFS based on the stochastic iteration of the two maps:

$$M_1(x)=\frac{x}{2}, \tag{63}$$

$$M_2(x)=\frac{x}{2}+\frac{1}{2}. \tag{64}$$

The attractor for M_1 is $x=0$, and the attractor for M_2 is $x=1$. This is most transparent in the binary representation,

$$x=\sum_{n=1}^{\infty}\frac{b_n}{2^n}, \tag{65}$$

where each b_n is either 0 or 1, and x has the binary decimal expansion: $0.b_1b_2b_3\cdots$. The action of the maps takes the form

$$M_1(x)=0.0b_1b_2b_3\cdots, \tag{66}$$

$$M_2(x)=0.1b_1b_2b_3\cdots. \tag{67}$$

Now, it is obvious that the attractor for the IFS based on these two maps is the entire real interval $[0,1]$, with measure 1 and [fractal!] dimension 1.[29] The equilibrium density, i.e. the attractor, is the constant density 1, just as for the Bernoulli map, Frobenius–Perron equation, but this time for a very different reason. In the Bernoulli case, instability expands a density's standard deviation until uniformity is reached, whereas in the IFS case, stable point attractors for the component maps generate a fractal dust of sharp points when the maps are iterated stochastically. In the special case of the maps in Eqs. (63) and (64), the fractal dust happens to be the continuum, $[0,1]$.

Another difference between the convergence of the Bernoulli map and the IFS to their respective equilibrium densities can be seen by noting that the Frobenius–Perron equation for the IFS is

$$P_{\mathrm{IFS}}f(x)=f(2x)\Theta(\tfrac{1}{2}-x)+f(2x-1)\Theta(x-\tfrac{1}{2}), \tag{68}$$

in which $\Theta(x)$ is the Heaviside theta function. This Frobenius–Perron operator is the adjoint[8–10,14] of the Frobenius–Perron operator for the Bernoulli map in Eq. (5). Convergence to equilibrium for the Bernoulli map is in the strong sense whereas for the IFS, it is in the weak sense,[14] again reflecting the different ways in which the equilibrium density is approached.

VI. THE BAKER MAP

The Baker map[3,8,9,30,31] is defined by

$$(x_{n+1},y_{n+1})=\begin{cases}\left(2x_n,\dfrac{y_n}{2}\right),\\[2ex] \left(2x_n-1,\dfrac{y_n}{2}+\dfrac{1}{2}\right),\end{cases} \tag{69}$$

wherein the upper expression holds for $0\leqslant x_n<\frac{1}{2}$, and the lower expression holds for $\frac{1}{2}\leqslant x_n\leqslant 1$. This map is area preserving and invertible, i.e. reversible. The Frobenius–Perron equation is

$$\begin{aligned}P_{\mathrm{Baker}}f(x,y)&=\int_0^{1/2}dx'\int_0^1 dy'\;\delta(x-2x')\,\delta\!\left(y-\frac{y'}{2}\right)\\&\quad\times f(x',y')+\int_{\frac{1}{2}}^1 dx'\int_0^1 dy'\;\delta[x-(2x'-1)]\\&\quad\times\delta\!\left[y-\left(\frac{y'}{2}+\frac{1}{2}\right)\right]f(x',y')\\&=\begin{cases}f\!\left(\dfrac{x}{2},2y\right),\\[2ex] f\!\left(\dfrac{x}{2}+\dfrac{1}{2},2y-1\right),\end{cases}\end{aligned} \tag{70}$$

wherein the upper expression holds for $0\leqslant y<\frac{1}{2}$, and the lower expression holds for $\frac{1}{2}\leqslant y\leqslant 1$.

The eigenstates and eigenvalue spectrum for the Baker map are more complicated than for the one-dimensional maps presented so far. The eigenvalues are again reciprocal powers of 2, but now there is also degeneracy that grows with the power.[8,9] The reader is referred to the references[8,9] for details regarding the construction of the eigenfunctions.

A *contraction of the description*[15] is achieved by integrating over y. This has been referred to as "coarse graining,"[3] but this is a misnomer. Let $g(x)$ be defined by the contraction[30,31]

$$g(x)=\int_0^1 dy\ f(x,y). \tag{71}$$

Therefore,

$$\begin{aligned} g_{n+1}(x) &= \int_0^1 dy\ P_{\text{Baker}} f_n(x,y) \\ &= \int_0^{1/2} dy\ f_n\left(\frac{x}{2},2y\right)+\int_{1/2}^1 dy\ f_n\left(\frac{x}{2}+\frac{1}{2},2y-1\right) \\ &= \frac{1}{2}\int_0^1 dy\ f_n\left(\frac{x}{2},y\right)+\frac{1}{2}\int_0^1 dy\ f_n\left(\frac{x}{2}+\frac{1}{2},y\right) \\ &= \frac{1}{2}\, g_n\left(\frac{x}{2}\right)+\frac{1}{2}\, g_n\left(\frac{x}{2}+\frac{1}{2}\right), \end{aligned} \tag{72}$$

where the third equality follows from simple changes of variable in each of the integrals. The result is the Frobenius–Perron equation for the Bernoulli map, Eq. (5).

Now, consider the action of P_{Baker} on two-dimensional periodic Gaussian distributions:

$$f_{\alpha,\beta}(x,y)=\frac{1}{2\pi\alpha\beta}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp\left(-\frac{(x-x_0+n)^2}{2\alpha^2}\right)\times\exp\left(-\frac{(y-y_0+m)^2}{2\beta^2}\right). \tag{73}$$

Clearly,

$$P_{\text{Baker}} f_{\alpha,\beta}(x,y)=\begin{cases} \dfrac{1}{2\pi\alpha\beta}\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp\left(-\frac{(x/2-x_0+n)^2}{2\alpha^2}\right)\exp\left(-\frac{(2y-y_0+m)^2}{2\beta^2}\right), \\ \dfrac{1}{2\pi\alpha\beta}\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp\left(-\frac{(x/2+\frac{1}{2}-x_0+n)^2}{2\alpha^2}\right)\exp\left(-\frac{(2y-1-y_0+m)^2}{2\beta^2}\right), \end{cases}$$

$$=\begin{cases} \dfrac{1}{2\pi\alpha\beta}\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp\left(-\frac{(x-2x_0+2n)^2}{2(2\alpha)^2}\right)\exp\left(-\frac{(y-y_0/2+m/2)^2}{2(\beta/2)^2}\right), \\ \dfrac{1}{2\pi\alpha\beta}\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\exp\left(-\frac{(x-2x_0+2n+1)^2}{2(2\alpha)^2}\right)\exp\left(-\frac{(y-y_0/2+m/2-\frac{1}{2})^2}{2(\beta/2)^2}\right), \end{cases} \tag{74}$$

wherein the upper expressions hold for $0\leqslant y<\frac{1}{2}$, and the lower expressions hold for $\frac{1}{2}\leqslant y\leqslant 1$. Note that $\alpha\beta=(2\alpha)(\beta/2)$ so that the normalizations match the variances in the exponentials. We may prove that normalization is preserved by noting that

$$\sum_{m=-\infty}^{\infty}\int_0^{1/2} dy\ \exp\left(-\frac{(y-y_0/2+m/2)^2}{2(\beta/2)^2}\right)=\sum_{m=-\infty}^{\infty}\int_{m/2}^{1/2+m/2} dy\ \exp\left(-\frac{(y-y_0/2)^2}{2(\beta/2)^2}\right) \tag{75}$$

and

$$\sum_{m=-\infty}^{\infty}\int_{1/2}^{1} dy\ \exp\left(-\frac{(y-y_0/2+m/2-\frac{1}{2})^2}{2(\beta/2)^2}\right)=\sum_{m=-\infty}^{\infty}\int_{m/2}^{1/2+m/2} dy\ \exp\left(-\frac{(y-y_0/2)^2}{2(\beta/2)^2}\right). \tag{76}$$

Therefore,

$$\begin{aligned} &\int_0^1 dx\int_0^1 dy\ P_{\text{Baker}} f_{\alpha,\beta}(x,y) \\ &= \frac{1}{2\pi\alpha\beta}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{m/2}^{1/2+m/2} dy\ \exp\left(-\frac{(y-y_0/2)^2}{2(\beta/2)^2}\right)\times\left[\int_{2n}^{2n+1} dx\ \exp\left(-\frac{(x-2x_0)^2}{2(2\alpha)^2}\right)+\int_{2n+1}^{2n+2} dx \right. \\ &\quad \left. \times\exp\left(-\frac{(x-2x_0)^2}{2(2\alpha)^2}\right)\right] \\ &= \frac{1}{\sqrt{[2\pi(\beta/2)^2]}}\int_{-\infty}^{\infty} dy\ \exp\left(-\frac{(y-y_0/2)^2}{2(\beta/2)^2}\right)\frac{1}{\sqrt{[2\pi(2\alpha)^2]}}\int_{-\infty}^{\infty} dx\ \exp\left(-\frac{(x-2x_0)^2}{2(2\alpha)^2}\right)=1. \end{aligned} \tag{77}$$

It is clear from Eq. (74) that iteration of $P_{\rm Baker}$ will cause the standard deviation for the x distribution to repeatedly double, thereby broadening the x distribution, whereas the standard deviation for the y distribution will repeatedly halve, thereby making the y distribution sharper and sharper. Asymptotically, the x distribution will become uniform, just as for the Bernoulli map. A stochastic interpretation of the x distribution converts the Baker map of Eq. (69) into

$$y_{n+1}=\begin{cases}\dfrac{y_n}{2}, \\ \dfrac{y_n}{2}+\dfrac{1}{2},\end{cases} \tag{78}$$

where each possibility occurs with probability $\frac{1}{2}$, which is precisely the IFS defined in Eqs. (63) and (64). This amounts to a *contraction of the description* in which the x variable is interpreted stochastically, rather than integrated.

VII. INTERPRETATION OF THE RESULTS

Two types of trajectory behavior have been elucidated by the analysis of the maps discussed in this paper. These behaviors exhibit features of the dynamics that go beyond the implications of exponential sensitivity to variations in initial conditions, the characterization usually invoked to define chaos in deterministic dynamical systems. One type of behavior, exemplified by the Bernoulli and Tent maps, exhibits the noncommutativity of the limits $\sigma\rightarrow 0$ and $n\rightarrow\infty$, whereas the other, exemplified by the Baker map, exhibits two distinct contractions of the description in which a deterministic invertible map is contracted into stochastic processes.

The Frobenius–Perron equations for the Bernoulli and Tent maps admit two types of solutions: δ functions and extended densities. In the text, densities characterized by the standard deviation σ have been represented by periodic Gaussian distributions on [0,1]. In the limit $\sigma\rightarrow 0$, these periodic Gaussian distributions become periodic δ functions that are solutions to the Frobenius–Perron equation and that precisely follow pointwise trajectory solutions to the Bernoulli or the Tent maps. In the limit $n\rightarrow\infty$, where n is the number of iterations, trajectory solutions chaotically explore the invariant attractor forever. However, for $\sigma>0$, the limit $n\rightarrow\infty$ changes an initially periodic Gaussian distribution into the uniform density on [0,1]. The noncommutativity of $\sigma\rightarrow 0$ and $n\rightarrow\infty$ means that *trajectories are unstable solutions*. Does this mean that pointwise trajectories cannot exist? If so, then ensembles of pointwise trajectories would no longer be meaningful. The stable extended density solution, $f_\sigma(x;x_0)$ which propagates as the periodic Gaussian $f_{2n_\sigma}[x;D_{\rm B}^{0n}(x_0)]$ in the case of the Bernoulli map, represents the putatively physical state. Can we maintain its ensemble interpretation in terms of pointwise trajectories or does it have a purely stochastic meaning?

It may be objected that the Bernoulli and Tent maps are not generic. Two features of the properties obtained above, in fact, are not generic: propagation as a periodic *Gaussian in x;* and, asymptotic approach to a *uniform density* in x. already, with the Tent map, we have seen that propagation as a periodic Gaussian is significantly modified [cf. Eq. (52)] as compared with the Bernoulli map [cf. Eq. (51)]. In general, propagation as a periodic *Gaussian in x* is only an approximation, and then only for as long as the growing standard deviation remains small compared to unity. For sufficiently small initial σ, this could still involve many orders of magnitude of growth. Also generally, the asymptotic invariant attractor is not uniform. An example is given by the Quadratic map,[3,14]

$$x_{n+1}=Q(x_n), \tag{79}$$

in which

$$Q(x)=4x(1-x). \tag{80}$$

Since the Quadratic map is topologically conjugate the Tent map,[14] the eigenstates and eigenvalues for this map can be constructed explicitly and are discussed in Appendix C. The Frobenius–Perron equation for this map has the invariant attractor density[3,14]

$$f(x)=\frac{1}{\pi\sqrt{[x(1-x)]}}, \tag{81}$$

which is clearly nonuniform. In addition, the local expansion rate varies with the position of the initial ensemble. Thus, the local and global Lyapunov exponents are no longer equal as they were for the Bernoulli and Tent maps. However, as is shown in Appendix C, the eigenvalues are identical to those for the Tent map, which were determined by simple multiples of the global Lyapunov exponent. Detailed results for the Quadratic map will be the subject of another paper. In the following, we show how to extend our analysis to the general case, which, of course, includes the Quadratic map.

Let a generic map on [0,1] be denoted by $M(x)$, so that

$$x_{n+1}=M(x_n). \tag{82}$$

Consider the associated Frobenius–Perron equation [cf. Eqs. (4) and (5)],

$$P_M f(x)=\sum_{\{M^{-1}(x)\}}\frac{f[M^{-1}(x)]}{|M'(M^{-1}(x))|}, \tag{83}$$

in which $\{M^{-1}(x)\}$ denotes the set of inverse images of x and M' is the derivative of M. Trajectory solutions to this equation exist that emanate from the $\sigma\rightarrow 0$ limit of

$$f_\sigma(x;x_0)=\frac{1}{\sqrt{(2\pi\sigma^2)}}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{(x-x_0+n)^2}{2\sigma^2}\right), \tag{84}$$

as the initial state. However, these trajectory solutions are unstable, as can be seen by considering the action of P_M on f_σ for $0<\sigma\ll 1$. For most $x_0\in[0,1]$, the $n=0$ term in Eq. (84) is overwhelmingly largest. (If not, i.e. for x_0 near 1 or 0, then the $n=-1$ or $+1$ terms need to be included [cf. Eq. (34)].) Therefore, we have, to very good approximation,

$$P_M f_\sigma(x;x_0) \approx P_M \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$
$$= \sum_{\{M^{-1}(x)\}} \frac{1}{\sqrt{(2\pi\{|M'[M^{-1}(x)]|\sigma\}^2)}} \times \exp\left(-\frac{[M^{-1}(x)-x_0]^2}{2\sigma^2}\right). \quad (85)$$

The last equality is exact. The resulting Gaussian is *not Gaussian in* x, but in $M^{-1}(x)$ instead. In the cases of the Bernoulli and Tent maps, this situation was easily remedied by multiplication by 2's, but in general no such simple manipulation is possible. Nevertheless, by invoking the method of steepest descent, we can acquire an excellent approximation, as long as σ is much less than unity.

The maximum, x^*, is located in accord with the vanishing of the first derivative of the argument of the exponential (the contribution from the x dependence in the denominator radical is negligible for $\sigma \ll 1$):

$$[M^{-1}(x^*)-x_0][M^{-1}(x^*)]' = 0, \quad (86)$$

which implies

$$x^* = M(x_0). \quad (87)$$

The second derivative, needed for the steepest descent approximation, is

$$\frac{1}{\sigma^2}\frac{d}{dx}\{[M^{-1}(x)-x_0][M^{-1}(x)]'\} = \frac{1}{\sigma^2}\{[M^{-1}(x^*)]'\}^2, \quad (88)$$

at $x = x^*$ because of Eq. (86). Because

$$M[M^{-1}(x)] = x, \quad (89)$$

we have

$$M'[M^{-1}(x)][M^{-1}(x)]' = 1, \quad (90)$$

or, from Eqs. (86) and (87),

$$[M^{-1}(x^*)]' = \frac{1}{M'[M^{-1}(x^*)]} = \frac{1}{M'(x_0)}. \quad (91)$$

Therefore, the steepest descent approximation is

$$\frac{1}{\sqrt{\{2\pi[|M'(M^{-1}(x))|\sigma]^2\}}} \exp\left(-\frac{[M^{-1}(x)-x_0]^2}{2\sigma^2}\right)$$
$$= \frac{1}{\sqrt{\{2\pi[|M'(x_0)|\sigma]^2\}}} \exp\left(-\frac{[x-M(x_0)]^2}{2[|M'(x_0)|\sigma]^2}\right). \quad (92)$$

The standard deviation has grown by $|M'(x_0)|$, which is precisely what one would expect from one iteration of the map if one is computing the Lyapunov exponent.[3] This is even more evident if we iterate again:

$$P_M^{02} f_\sigma(x;x_0) \approx P_M^{02} \frac{1}{\sqrt{(2\pi\sigma^2)}} \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)$$
$$= \sum_{\{M^{-1}[M^{-1}(x)]\}} \sum_{\{M^{-1}(x)\}} \frac{1}{\sqrt{(2\pi(|M'\{M^{-1}[M^{-1}(x)]\}||M'[M^{-1}(x)]|\sigma)^2)}} \times \exp\frac{\{M^{-1}[M^{-1}(x)]-x_0\}^2}{2\sigma^2}. \quad (93)$$

Again, the last equality is exact. There is also a second summation over inverse images of inverse images. It should now be evident what occurs for an application of P_M^{0n}. The steepest descent approximation requires equations paralleling Eqs. (86) and (88), which are

$$\{M^{-1}[M^{-1}(x^{**})]-x_0\}\{M^{-1}[M^{-1}(x^{**})]\}' = 0 \quad (94)$$

and

$$\frac{1}{\sigma^2}\frac{d}{dx}(\{M^{-1}[M^{-1}(x)]-x_0\}\{M^{-1}[M^{-1}(x)]\}')$$
$$= \frac{1}{\sigma^2}(\{M^{-1}[M^{-1}(x^{**})]\}')^2, \quad (95)$$

at $x = x^{**}$, where

$$x^{**} = M^{02}(x_0). \quad (96)$$

The chain rule implies

$$\{M^{-1}[M^{-1}(x)]\}' = \{(M^{-1})'[M^{-1}(x)]\}[M^{-1}(x)]'. \quad (97)$$

From

$$x = M^{02}\{M^{-1}[M^{-1}(x)]\}, \quad (98)$$

differentiation yields

$$1 = M'(M\{M^{-1}[M^{-1}(x)]\})M'\{M^{-1}[M^{-1}(x)]\} \times \{(M^{-1})'[M^{-1}(x)]\}[M^{-1}(x)]'$$
$$= M'[M^{-1}(x)]M'\{M^{-1}[M^{-1}(x)]\} \times \{(M^{-1})'[M^{-1}(x)]\}[M^{-1}(x)]'. \quad (99)$$

Equation (90) implies a partial cancellation, giving

$$\{(M^{-1})'[M^{-1}(x)]\} = \frac{1}{M'\{M^{-1}[M^{-1}(x)]\}}. \quad (100)$$

Using this and Eq. (90) on the right-hand side of Eq. (97) gives

$$\{M^{-1}[M^{-1}(x)]\}' = \frac{1}{M'\{M^{-1}[M^{-1}(x)]\}} \frac{1}{M'[M^{-1}(x)]}. \tag{101}$$

Thus, the steepest descent approximation to the right-hand side of Eq. (93) is

$$\frac{1}{\sqrt{[2\pi(|M'\{M^{-1}[M^{-1}(x)]\}||M'[M^{-1}(x)]|\sigma)^2]}} \times \exp\left(-\frac{\{M^{-1}[M^{-1}(x)]-x_0\}^2}{2\sigma^2}\right) \approx \frac{1}{\sqrt{(2\pi\{|M'[M(x_0)]||M'(x_0)|\sigma\}^2)}} \times \exp\left(-\frac{[x-M^{02}(x_0)]^2}{2\{|M'[M(x_0)]||M'(x_0)|\sigma\}^2}\right). \tag{102}$$

The standard deviation has grown by

$$M'[M(x_0)]M'(x_0)=M'(x_1)M'(x_0). \tag{103}$$

Clearly, after n iterations, this generalizes to

$$\sigma \to \prod_{j=0}^{n-1} M'(x_j)\sigma, \tag{104}$$

in which x_j is the jth iterate emanating from x_0. This is consistent with the formula for the global Lyapunov exponent for a map,[3]

$$\lambda_L = \underset{n\to\infty}{\text{limit}} \frac{1}{n} \ln \prod_{j=0}^{n-1} M'(x_j). \tag{105}$$

Of course, in Eq. (104) we cannot let $n\to\infty$ because once the standard deviation approaches unity, the steepest descent argument breaks down and many more terms in Eq. (84) become important, having developed long range tails that extend into [0,1]. Nevertheless, we have approximately Gaussian propagation, and the rate of growth of the standard deviation given in Eq. (104) for finite n is the obvious ingredient for the quantitative concept of a "local," transient, Lyapunov exponent[28] [cf. Eq. (105) for finite n]. This is identical with the idea of a local Lyapunov exponent that we used in our work on quantum signatures of classical chaos.[20,21]

We may understand the existence of nonuniform asymptotic, invariant densities in terms of the periodic Gaussian distributions as well. We need not restrict ourselves to the $n=0$ term in Eq. (84). Clearly, we can write the exact equality,

$$P_M f_\sigma(x;x_0) = \sum_{\{M^{-1}(x)\}} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{(2\pi\{|M'[M^{-1}(x)]|\sigma\}^2)}} \times \exp\left(-\frac{[M^{-1}(x)-x_0+n]^2}{2\sigma^2}\right). \tag{106}$$

Further iteration of P_M acts as illustrated by P_M^{02} in Eq. (93). At iteration n, we obtain a summation of periodic Gaussians over sets of higher-order inverse images of x. In the limit $n\to\infty$, the proliferation of inverse images creates the nonuniform, invariant density. The uniform density occurs only when these Gaussians, in infinite-order inverse images of x, can be manipulated into Gaussians in x, a possibility only in special cases, e.g. the Bernoulli and Tent maps.

Instability of pointwise trajectory solutions appears to be generic for chaotic maps. Any initially extended density will irreversibly approach an invariant, not necessarily uniform, measure on the attractor. Do we view this extended density as an ensemble of points, or simply as a probability distribution?

The Baker map, Frobenius–Perron equation, Eq. (70), permits two contractions of the description. The well-known one is to integrate out the y variable, which produces the Frobenius–Perron equation for the Bernoulli map, as was shown in Sec. VI. This explains why the instability of trajectory solutions for the Baker map is essentially the same as that for the Bernoulli map. In this case, contraction of the description plays no role in creating the instability or its associated irreversibility. The other contraction involves a stochastic reinterpretation of the extended nature of the x variable density as it approaches an invariant measure as a result of instability. This reinterpretation produces an IFS for the y variable. In the ensemble interpretation, we have many unstable pointwise trajectories, and the IFS obtained is for the behavior of y, averaged over the ensemble. In the stochastic interpretation, the x density is a probability distribution for x and we have an IFS for the y variable.

Whichever interpretation is given to extended densities, they manifest intrinsically irreversible evolution for dynamical systems, e.g. the Baker map, that show fully reversible pointwise trajectory evolution. The rates of approach to the extended asymptotic stationary states and the positive, "local" Lyapunov exponents [cf. the discussion following Eq. (105)] are both determined by the eigenvalues of the Frobenius–Perron operator, or equivalently by the Pollicott–Ruelle resonances.[12] In the cases examined here, the eigenvalues have been seen to be simply related to the global Lyapunov exponent. It remains to be seen how general this feature is as a broader class of maps is studied.

The time evolution of pointwise trajectories is unstable. It seems natural to view ensembles of unstable trajectories as probability distributions for single extended states.

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APPENDIX A: BERNOULLI MAP EIGENFUNCTIONS

The Bernoulli polynomials are defined symbolically[34] by

$$B_m(x)=(x+B)^m=\sum_{n=0}^{m}\binom{m}{n}B_n x^{m-n}, \tag{A1}$$

in which the symbolic replacement, $B^n\to B_n$, is made and B_n is the nth Bernoulli number. The Bernoulli numbers in turn are defined symbolically[34] by

$$(1+B)^n-B_n=0, \quad B_0=1, \; n>1. \tag{A2}$$

The first few equations following from Eq. (A2) for $n=2,3,4,...,$ are

$$1+2B_1=0, \tag{A3}$$

$$1+3B_1+3B_2=0, \tag{A4}$$

$$1+4B_1+6B_2+4B_3=0, \tag{A5}$$

from which it follows that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, etc. From Eq. (A1), it is clear that $B_m(0)=B_m$, and from Eqs. (A1) and (A2) it is also clear that $B_m(1)=B_m$ as well, as long as $m>1$. For $m=0$, $B_0(x)=1=B_0$, which is consistent with these identities, but for $m=1$, $B_1(x)=x-\frac{1}{2}$ from Eqs. (A1) and (A3), so that $B_1(1)=\frac{1}{2}$, whereas $B_1(0)=-\frac{1}{2}$. The first few $B_m(x)$'s are

$$B_0(x)=1, \tag{A6}$$

$$B_1(x)=x-\tfrac{1}{2}, \tag{A7}$$

$$B_2(x)=x^2-x+\tfrac{1}{6}, \tag{A8}$$

$$B_3(x)=x^3-\tfrac{3}{2}x^2+\tfrac{1}{2}x. \tag{A9}$$

Equation (A1) implies a generating function for the $B_m(x)$'s:

$$\sum_{m=0}^{\infty} B_m(x)\,\frac{t^m}{m!}=\sum_{m=0}^{\infty}(x+B)^m\,\frac{t^m}{m!} = \exp[t(x+B)]= \exp[tx]\exp[tB]. \tag{A10}$$

However,

$$\begin{aligned}\exp[tB]&=1+tB_1+\sum_{m=2}^{\infty}B_m\,\frac{t^m}{m!}\\ &=1-\frac{t}{2}+\sum_{m=2}^{\infty}(1+B)^m\,\frac{t^m}{m!}\\ &=1-\frac{t}{2}+\{\exp[t(1+B)]-1-t(1+B_1)\}\\ &=-t+\exp[t(1+B)].\end{aligned} \tag{A11}$$

Therefore,

$$\exp[tB](1-\exp[t])=-t \tag{A12}$$

or

$$\exp[tB]=\frac{t}{\exp[t]-1}. \tag{A13}$$

Inserting this into Eq. (A10) yields the generating function,

$$\frac{t\,\exp[xt]}{\exp[t]-1}=\sum_{m=0}^{\infty}B_m(x)\,\frac{t^m}{m!}. \tag{A14}$$

From Eq. (A14), differentiation with respect to x yields

$$\frac{t^2\,\exp[xt]}{\exp[t]-1}=\sum_{m=0}^{\infty}\frac{d}{dx}B_m(x)\,\frac{t^m}{m!}=\sum_{m=1}^{\infty}\frac{d}{dx}B_m(x)\,\frac{t^m}{m!}. \tag{A15}$$

Division by t produces

$$\begin{aligned}\frac{t\,\exp[xt]}{\exp[t]-1}&=\sum_{m=1}^{\infty}\frac{d}{dx}B_m(x)\,\frac{t^{m-1}}{m!}\\ &=\sum_{m=0}^{\infty}\frac{d}{dx}B_{m+1}(x)\,\frac{t^m}{(m+1)!}.\end{aligned} \tag{A16}$$

Together with Eq. (A14), this implies

$$\frac{d}{dx}B_{m+1}(x)=(m+1)B_m(x). \tag{A17}$$

Integration produces

$$\begin{aligned}B_{m+1}(x)&=(m+1)\int_0^x dy\;B_m(y)+B_{m+1}(0)\\ &=(m+1)\int_0^x dy\;B_m(y)+B_{m+1}.\end{aligned} \tag{A18}$$

This enables us to generate all $B_m(x)$'s from $B_0(x)=1$ and the Bernoulli numbers. Another useful identify follows from Eq. (A17):

$$\begin{aligned}\int_0^1 dx\;B_m(x)&=\frac{1}{m+1}\int_0^1 dx\,\frac{d}{dx}B_{m+1}(x)\\ &=\frac{1}{m+1}[B_{m+1}(1)-B_{m+1}(0)]\\ &=0,\quad \text{for}\quad m\geqslant 1,\end{aligned} \tag{A19}$$

where the last line follows from the remarks between Eqs. (A5) and (A6).

APPENDIX B: TENT MAP EIGENFUNCTIONS

The Frobenius–Perron equation for the Tent map given in Eq. (39) has eigenstate solutions satisfying Eqs. (53) and (54).[38] In this appendix, we give an elementary derivation of these equations.

From Eq. (A14), it follows that

$$\frac{t\,\exp[(1-x)t]}{\exp[t]-1}=\sum_{m=0}^{\infty}B_m(1-x)\,\frac{t^m}{m!}. \tag{B1}$$

Consequently,

$$P_{\mathrm{T}}\,\frac{t\,\exp[xt]}{\exp[t]-1}=\frac{1}{2}\,\frac{t}{\exp[t]-1}\left(\exp\left[\frac{x}{2}\,t\right]+\exp\left[\left(1-\frac{x}{2}\right)t\right]\right) \tag{B2}$$

and

$$P_{\mathrm{T}}\,\frac{t\,\exp[(1-x)t]}{\exp[t]-1}=\frac{1}{2}\,\frac{t}{\exp[t]-1}\left(\exp\left[\left(1-\frac{x}{2}\right)t\right]+\exp\left[1-\left(1-\frac{x}{2}\right)t\right]\right). \tag{B3}$$

Since the right-hand sides are equal, subtraction of Eq. (B3) from Eq. (B2) gives

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} P_\mathrm{T}[B_m(x)-B_m(1-x)]=0. \tag{B4}$$

Now, replace t by $-t$ on the left-hand side of Eq. (B1):

$$\frac{-t\ \exp[(x-1)t]}{\exp[-t]-1}=\frac{t\ \exp[xt]}{\exp[t]-1}=\sum_{m=0}^{\infty} \frac{t^m}{m!} B_m(x), \tag{B5}$$

and do the same for the right-hand side, which, together with Eq. (B5), yields

$$\sum_{m=0}^{\infty} \frac{t^m}{m!} (-1)^m B_m(1-x)=\sum_{m=0}^{\infty} \frac{t^m}{m!} B_m(x), \tag{B6}$$

from which we conclude that

$$B_m(1-x)=(-1)^m B_m(x). \tag{B7}$$

Using this identity in Eq. (B4) produces the eigenfunction equation for odd indices, Eq. (54), when $ET_{2m+1}(x)$, is defined by

$$ET_{2m+1}(x)=B_{2m+1}(x). \tag{B8}$$

In order to obtain the eigenfunction equation for even indices, Eq. (53), we need the following variations on Eq. (A14) and Eq. (B1):

$$\frac{t\ \exp[(x/2)t]}{\exp[t]-1}=\sum_{m=0}^{\infty} B_m\left(\frac{x}{2}\right) \frac{t^m}{m!} \tag{B9}$$

and

$$\frac{t\ \exp[(1-x/2)t]}{\exp[t]-1}=\sum_{m=0}^{\infty} B_m(1-x/2) \frac{t^m}{m!}. \tag{B10}$$

Variations on Eqs. (B2) and (B3) give

$$\begin{aligned}P_\mathrm{T} \frac{t\ \exp[(x/2)t]}{\exp[t]-1} &= \frac{1}{2} \frac{t}{\exp[t]-1} \left(\exp\left[\frac{x}{4} t\right]+\exp\left[\frac{1}{2}\left(1-\frac{x}{2}\right)t\right]\right) \\ &= \frac{1}{2} \frac{t}{\exp[t]-1} \left(\exp\left[\frac{x}{4} t\right]+\exp\left[\frac{t}{2}\right]\exp\left[-\frac{x}{4} t\right]\right)\end{aligned} \tag{B11}$$

and

$$\begin{aligned}P_T \frac{t\ \exp[(1-x/2)t]}{\exp[t]-1} &= \frac{1}{2} \frac{t}{\exp[t]-1} \left(\exp\left[\left(1-\frac{x}{4}\right)t\right] + \exp\left[1-\frac{1}{2}\left(1-\frac{x}{2}\right)t\right]\right) \\ &= \frac{1}{2} \frac{t\ \exp[t/2]}{\exp[t]-1} \left(\exp\left[\frac{x}{4} t\right] + \exp\left[\frac{t}{2}\right]\exp\left[-\frac{x}{4} t\right]\right).\end{aligned} \tag{B12}$$

Adding these two equations and substituting the right-hand sides of Eqs. (B9) and (B10) gives

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{t^m}{m!} P_\mathrm{T}\left[B_m\left(\frac{x}{2}\right)+B_m\left(1-\frac{x}{2}\right)\right] &= \frac{t/2}{\exp[t/2]-1} \left(\exp\left[\frac{x}{2}\frac{t}{2}\right]+\exp\left[\left(1-\frac{x}{2}\right)\frac{t}{2}\right]\right) \\ &= \sum_{m=0}^{\infty} B_m\left(\frac{x}{2}\right) \frac{(t/2)^m}{m!} + \sum_{m=0}^{\infty} B_m\left(1-\frac{x}{2}\right) \frac{(t/2)^m}{m!}.\end{aligned} \tag{B13}$$

Replacing x by $x/2$ in the identity Eq. (B7), permits the deduction from Eq. (B13) that

$$P_\mathrm{T} B_{2m}\left(\frac{x}{2}\right)=\frac{1}{2^{2m}} B_{2m}\left(\frac{x}{2}\right). \tag{B14}$$

This is precisely the eigenfunction equation for even indices, Eq. (53), if $ET_{2m}(x)$ is defined by

$$ET_{2m}(x)=B_{2m}\left(\frac{x}{2}\right). \tag{B15}$$

The analog of Eq. (25) is

$$f(x)=\sum_{m=0}^{\infty} d_{2m} ET_{2m}(x). \tag{B16}$$

We restrict the indices in Eq. (B16) to even values because the action of P_T on odd indexed eigenstates is to annihilate them. To show that the expansion coefficients, d_{2m}, are unique only requires the parallel to Eq. (27),

$$\int_0^1 dx\ ET_m(x)=\delta_{m0}, \tag{B17}$$

and the slight variation of Eq. (A17):

$$\frac{d^2}{dx^2} ET_{2m}(x)=\frac{2m(2m-1)}{2^2} ET_{2m-2}(x). \tag{B18}$$

For odd index m, Eq. (B17) is trivial because of Eq. (B8). For even index m, we obtain

$$\int_0^1 dx\ ET_{2m}(x)=\int_0^1 dx\ B_{2m}\left(\frac{x}{2}\right)=2\int_0^{1/2} dx\ B_{2m}(x). \tag{B19}$$

From Eqs. (27) and (B7) we obtain

$$\begin{aligned}\delta_{m0} &= \int_0^1 dx\ B_{2m}(x) \\ &= \int_0^{1/2} dx\ B_{2m}(x)+\int_{1/2}^1 dx\ B_{2m}(x) \\ &= \int_0^{1/2} dx\ B_{2m}(x)+\int_0^{1/2} dx\ B_{2m}(1-x) \\ &= 2\int_0^{1/2} dx\ B_{2m}(x),\end{aligned} \tag{B20}$$

which concludes the proof of Eq. (B17). From Eqs. (B18) and (B20), and an argument like that used to obtain Eq. (26), we obtain the analog to Eq. (33),

$$d_{2n}=\frac{2^{2n}}{(2n)!}\,[f^{(2n-1)}(1)-f^{(2n-1)}(0)]. \tag{B21}$$

APPENDIX C: QUADRATIC MAP EIGENFUNCTIONS

The Frobenius–Perron equation for the Quadratic map defined by Eq. (80) is[14]

$$P_{\mathrm{Q}}f(x)=\frac{1}{4\sqrt{(1-x)}}\left[f\left(\frac{1}{2}-\frac{1}{2}\sqrt{(1-x)}\right)+f\left(\frac{1}{2}+\frac{1}{2}\sqrt{(1-x)}\right)\right]. \tag{C1}$$

Let $g(x)$, which is monotone on [0,1], be defined by

$$g(x)=\frac{1}{2}-\frac{1}{\pi}\sin^{-1}(1-2x), \tag{C2}$$

which has the inverse

$$g^{-1}(x)=\frac{1}{2}-\frac{1}{2}\cos(\pi x). \tag{C3}$$

It has been shown[14] that

$$D_{\mathrm{T}}(x)=g\circ Q\circ g^{-1}(x), \tag{C4}$$

the symbol $\circ$ denotes functional composition. This implies[14]

$$P_gP_{\mathrm{Q}}f(x)=P_{\mathrm{T}}P_gf(x), \tag{C5}$$

in which P_g is the Frobenius–Perron operator based on the function $g(x)$ and is given by

$$P_gf(x)=\frac{\pi\sin(\pi x)}{2}\,f\left(\frac{1}{2}-\frac{1}{2}\cos(\pi x)\right). \tag{C6}$$

Now choose

$$f(x)=(P_g)^{-1}B_{2m}\left(\frac{x}{2}\right). \tag{C7}$$

Inserting this into Eq. (C5) yields

$$P_gP_{\mathrm{Q}}(P_g)^{-1}B_{2m}\left(\frac{x}{2}\right)=P_{\mathrm{T}}B_{2m}\left(\frac{x}{2}\right)=\frac{1}{2^{2m}}\,B_{2m}\left(\frac{x}{2}\right), \tag{C8}$$

by Eq. (B14). Applying $(P_g)^{-1}$ to both sides yields the Quadratic map eigenfunction–eigenvalue equation,

$$P_{\mathrm{Q}}(P_g)^{-1}B_{2m}\left(\frac{x}{2}\right)=\frac{1}{2^{2m}}\,(P_g)^{-1}B_{2m}\left(\frac{x}{2}\right). \tag{C9}$$

This shows that the spectrum is invariant under topological conjugacy. It is also easy to show that

$$(P_g)^{-1}f(x)=P_{g^{-1}}f(x), \tag{C10}$$

and that

$$P_{g^{-1}}f(x)=\frac{1}{\pi\sqrt{[x(1-x)]}}\,f\left(\frac{1}{2}-\frac{1}{\pi}\sin^{-1}(1-2x)\right). \tag{C11}$$

Consequently, the eigenfunctions for P_{Q} are

$$(P_g)^{-1}B_{2m}\left(\frac{x}{2}\right)=\frac{1}{\pi\sqrt{[x(1-x)]}}\times B_{2m}\left(\frac{1}{4}-\frac{1}{2\pi}\sin^{-1}(1-2x)\right). \tag{C12}$$

We have also obtained the analog to Eq. (33) and Eq. (B21) but will present this result and a detailed numerical study for the Quadratic map in another paper.

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